MATCHINGS IN k-PARTITE k-UNIFORM HYPERGRAPHS

JIE HAN, CHUANYUN ZANG, AND YI ZHAO

Abstract. For $k \geq 3$ and $\epsilon > 0$, let H be a k-partite k-graph with parts $V_1, \dots, V_k$ each of size n, where n is sufficiently large. Assume that for each $i \in [k]$, every $(k-1)$-set in $\prod_{j \in [k] \setminus \{i\}} V_i$ lies in at least a_i edges, and $a_1 \geq a_2 \geq \cdots \geq a_k$. We show that if $a_1, a_2 \geq \epsilon n$, then H contains a matching of size $\min\{n-1, \sum_{i \in [k]} a_i\}$. In particular, H contains a matching of size $n-1$ if each crossing $(k-1)$-set lies in at least $\lceil n/k \rceil$ edges, or each crossing $(k-1)$-set lies in at least $\lfloor n/k \rfloor$ edges and $n \equiv 1 \bmod k$. This special case answers a question of Rödl and Ruciński and was independently obtained by Lu, Wang, and Yu.

The proof of Lu, Wang, and Yu closely follows the approach of Han [Combin. Probab. Comput. 24 (2015), 723–732] by using the absorbing method and considering an extremal case. In contrast, our result is more general and its proof is thus more involved: it uses a more complex absorbing method and deals with two extremal cases.

1. Introduction

A *k-uniform hypergraph* (in short, *k-graph*) consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$, that is, every edge is a k-element subset of V. A k-graph H is *k-partite* if $V(H)$ can be partitioned into k parts $V_1, \dots, V_k$ such that every edge consists of exactly one vertex from each class, in other words, $E(H) \subseteq V_1 \times \cdots \times V_k$. A *matching* in H is a collection of vertex-disjoint edges of H. A matching covering all vertices of H is called *perfect*.

Given a k-graph H and a set S of d vertices in $V(H)$, where $1 \leq d \leq k-1$, a *neighbor* of S is a $(k-d)$-set $T \subseteq V(H) \setminus S$ such that $S \cup T \in E(H)$. Denote by $N_H(S)$ the set of the neighbors of S, and define the *degree* of S to be $\deg_H(S) = |N_H(S)|$. We omit the subscript H if it is clear from the context. The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-subsets S of $V(H)$. The minimum $(k-1)$-degree is also called the *minimum codegree*.

The minimum d-degree thresholds that force a perfect matching in k-graphs have been studied intensively, see [2, 3, 5, 8, 12, 13, 15, 18, 19, 22, 23, 24, 25, 26] and surveys [20, 29]. In particular, Rödl, Ruciński and Szemerédi [23] determined the minimum codegree threshold that guarantees a perfect matching in an n-vertex k-graph for large n and all $k \geq 3$. The threshold is $n/2 - k + C$, where $C \in \{3/2, 2, 5/2, 3\}$ depending on the values of n and k. In contrast, the minimum codegree threshold for a matching of size $\lceil n/k \rceil - 1$ is much smaller. Rödl, Ruciński and Szemerédi [23] showed that every k-graph H on n vertices satisfying $\delta_{k-1}(H) \geq n/k + O(\log n)$ contains a matching of size $\lceil n/k \rceil - 1$. Han [6] improved this by reducing the assumption to $\delta_{k-1}(H) \geq \lceil n/k \rceil - 1$, which is best possible.

In this paper we are interested in the corresponding thresholds in k-partite k-graphs. Suppose H is a k-partite k-graph with parts $V_1, \dots, V_k$. A subset $S \subset V(H)$ is called *crossing* if $|S \cap V_i| \leq 1$ for all i. For any $I \subseteq [k]$, let $\delta_I(H)$ be the minimum of $\deg_H(S)$ taken over all crossing $|I|$-vertex sets S in $\prod_{i \in I} V_i$. Then the *partite minimum d-degree* $\delta'_d(H)$ is defined as the minimum of $\delta_I(H)$ over all d-element sets $I \subseteq [k]$.

Let H be a k-partite k-graph with n vertices in each part. For $k \geq 3$, Kühn and Osthus [14] proved that if $\delta'_{k-1}(H) \geq n/2 + \sqrt{2n \log n}$ then H has a perfect matching. Later Aharoni, Georgakopoulos and Sprüssel [1] improved this result by requiring only two partite minimum codegrees. They showed that H contains a perfect matching if $\delta_{[k]\setminus\{1\}}(H) > n/2$ and $\delta_{[k]\setminus\{2\}}(H) \geq n/2$, and consequently, if $\delta'_{k-1}(H) > n/2$ then H has a perfect matching.

Similarly to the non-partite case, when targeting almost perfect matchings, the minimum degree threshold also drops significantly. Kühn and Osthus in [14] proved that $\delta'_{k-1}(H) \geq \lceil n/k \rceil$ guarantees a matching of size

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$n-(k-2)$. Rödl and Ruciński [20, Problem 3.14] asked whether $\delta'_{k-1}(H) \ge \lceil n/k \rceil$ guarantees a matching in H of size $n-1$. In this paper, we answer this question in the affirmative and show that the threshold can be actually weakened to $\lfloor n/k \rfloor$ if $n \equiv 1 \pmod{k}$. In fact, our result is much more general – it only requires that the sum of the partite minimum codegrees is large and at least two partite codegrees are not small.

Theorem 1.1 (Main Result). *For any $k \ge 3$ and $\epsilon > 0$, there exists n_0 such that the following holds for all $n \ge n_0$. Let H be a k-partite k-graph with parts of size n and $a_i := \delta_{[k]\setminus\{i\}}(H)$ for all $i \in [k]$ such that $a_1 \ge a_2 \ge \cdots \ge a_k$ and $a_2 > \epsilon n$. Then H contains a matching of size at least* $\min\{n-1, \sum_{i=1}^k a_i\}$.

Our proof, based on the absorbing method, unfortunately fails when a_1 is close to n and all of $a_2, \dots, a_k$ are small. It is unclear (to us) if the same assertion holds in this case.

The following corollary follows from Theorem 1.1 immediately. It was announced at [28] and appeared in the dissertation of the second author [27]. The second case of Corollary 1.2 resolves [20, Problem 3.14] and was independently proven by Lu, Wang and Yu [17].

Corollary 1.2. *Given $k \ge 3$, there exists n_0 such that the following holds for all $n \ge n_0$. Let H be a k-partite k-graph with parts of size n. Then H contains a matching of size $n-1$ if one of the following holds.*

- *$n \equiv 1 \bmod k$ and $\delta'_{k-1}(H) \ge \lfloor n/k \rfloor$;*
- *$\delta'_{k-1}(H) \ge \lceil n/k \rceil$.*

Let $\nu(H)$ be the size of a maximum matching in H. The following greedy algorithm, which essentially comes from [14], gives a simple proof of Theorem 1.1 when $\sum_{i=1}^k a_i \le n-k+2$ or when $a_1 + a_2 \ge n-1$.

Fact 1.3. *Let $n \ge k-2$. Suppose H is a k-partite k-graph with parts of size at least n. Let $a_i := \delta_{[k]\setminus\{i\}}(H)$ for $i \in [k]$. Then*

$$\nu(H) \ge \min\left\{n-k+2, \sum_{i=1}^k a_i\right\} \quad \textit{and} \quad \nu(H) \ge \min\{n-1, \, a_1 + a_2\}.$$

Proof. Assume a maximum matching M of H has size $|M| \le \min\{n-k+1, \sum_{i=1}^k a_i - 1\}$. Since each class has at least $k-1$ vertices unmatched, we can find k disjoint crossing $(k-1)$-sets $U_1, \dots, U_k$ such that U_i contains exactly one unmatched vertex in V_j for $j \ne i$. Each U_i has at least a_i neighbors and all of them lie entirely in $V(M)$. Since $\sum_{i=1}^k a_i > |M|$, there exist distinct indices $i \ne j$ such that U_i and U_j have neighbors on the same edge $e \in M$, say $v_i \in N(U_i) \cap e$ and $v_j \in N(U_j) \cap e$. Replacing e by $\{v_i\} \cup U_i$ and $\{v_j\} \cup U_j$ gives a larger matching, a contradiction. The second inequality can be proved similarly. $\square$

The following construction, sometimes called a *space barrier*, shows that the degree sum conditions in Theorem 1.1 and Fact 1.3 are best possible. Let $H_0 = H_0(a_1, \dots, a_k)$ be a k-partite k-graph with n vertices in each part $V_1, \dots, V_k$. For each $i \in [k]$, fix a set $A_i \subseteq V_i$ of size a_i. Let $E(H_0)$ consist of all crossing k-sets e such that $e \cap A_i \ne \emptyset$ for some $i \in [k]$. Suppose $\sum_{i=1}^k a_i \le n-1$. Clearly both $\nu(H_0)$ and the partite degree sum of H_0 equal to $\sum_{i=1}^k a_i$ (so we cannot expect a matching larger than $\sum_{i=1}^k a_i$).

Given a set V, let $V_1 \cup \cdots \cup V_k$ and $A \cup B$ be two partitions of V. For $i \in [k]$ we always write $A_i := A \cap V_i$ and $B_i := B \cap V_i$. A set $S \subseteq V$ is *even* (otherwise *odd*) if it intersects A in an even number of vertices. Let $E_{even}(A, B)$ (respectively, $E_{odd}(A, B)$) denote the family of all crossing k-subsets of V that are even (respectively, odd).

To see that we cannot always expect a perfect matching when $\sum_{i=1}^k a_i \ge n$, consider the following example, sometimes called a *divisibility barrier*. Let H_1 be a k-partite k-graph with n vertices in each of its parts $V_1, \dots, V_k$. For $i \in [k]$, let $V_i = A_i \cup B_i$ such that $\sum_{i=1}^k |A_i|$ is odd and for each $i \in [k]$, $n/2-1 \le |A_i| \le n/2+1$. Let $E(H_1) = E_{even}(A, B)$. So the partite degree sum of H_1 is at least $k(n/2-1)$. However, H_1 does not contain a perfect matching because any matching in H_1 covers an even number of vertices in $\bigcup_{i=1}^k A_i$ but $|\bigcup_{i=1}^k A_i|$ is odd.

When proving Corollary 1.2 directly, the authors of [17, 27] closely followed the approach used by the first author [6] by separating two cases based on whether H is close to H_0. In contrast, to prove Theorem 1.1, we have to consider *three* cases separately: when H is close to H_0, when H is close to (a weaker form of) H_1, and when H is far from both H_0 and H_1.

Now we define two extremal cases formally. Let H be a k-partite k-graph with each part of size n and let $a_i := \delta_{[k]\setminus\{i\}}(H)$ for all $i \in [k]$. We call H *ϵ-S-extremal* if $\sum_{i=1}^k a_i \le (1+\epsilon)n$ and $V(H)$ contains an independent set C such that $|C \cap V_i| \ge n - a_i - \epsilon n$ for each $i \in [k]$. We call H *ϵ-D-extremal* if there is a partition $A \cup B$ of $V(H)$ such that

(i) $(1/2-\epsilon)n \le a_1, a_2, |A_1|, |A_2| \le (1/2+\epsilon)n$, and $a_i \le \epsilon n$ for $i \ge 3$,
(ii) $|E_{even}(A,B) \setminus E(H)| \le \epsilon n^k$ or $|E_{odd}(A,B) \setminus E(H)| \le \epsilon n^k$.

Our proof of Theorem 1.1 consists of the following three theorems.

Throughout the paper, we write $\alpha \ll \beta \ll \gamma$ to mean that we can choose the positive constants α, β, γ from right to left. More precisely, there are increasing functions f and g such that, given γ, whenever $\beta \le f(\gamma)$ and $\alpha \le g(\beta)$, the subsequent statement holds. Hierarchies of other lengths are defined similarly. Moreover, when we use variables of the reciprocal form in the hierarchy, we implicitly assume that the variables are integers. Throughout this paper, we omit the assumption $k \ge 3$ in the hierarchies.

Theorem 1.4 (Non-extremal case). *Let $1/n \ll \gamma \ll \epsilon \ll 1/k$. Suppose H is a k-partite k-graph with parts of size n with $a_i := \delta_{[k]\setminus\{i\}}(H)$ for $i \in [k]$ such that $(1-\epsilon)n \ge a_1 \ge a_2 \ge \cdots \ge a_k$, $a_2 \ge \epsilon n$, and $\sum_{i=1}^k a_i \ge (1-\gamma/5)n$. Then one of the following holds:*

(i) *H contains a matching of size at least $n-1$;*
(ii) *H is γ-S-extremal;*
(iii) *H is $2k^2\epsilon$-D-extremal.*

Theorem 1.5 (Extremal case I). *Let $1/n \ll \gamma \ll \epsilon \ll 1/k$. Let H be a k-partite k-graph with parts of size n and $a_i := \delta_{[k]\setminus\{i\}}(H)$ for $i \in [k]$ such that $(1-\epsilon)n \ge a_1 \ge a_2 \ge \cdots \ge a_k$ and $a_2 \ge \epsilon n$. Suppose H is γ-S-extremal. Then one of the following holds:*

(i) *H contains a matching of size at least* $\min\{n-1, \sum_{i=1}^k a_i\}$*;*
(ii) *H is 3ϵ-D-extremal.*

Theorem 1.6 (Extremal case II). *Let $1/n \ll \epsilon \ll 1/k$. Suppose H is a k-partite k-graph with parts of size n and $a_i := \delta_{[k]\setminus\{i\}}(H)$ for $i \in [k]$. If H is ϵ-D-extremal, then H contains a matching of size at least* $\min\{n-1, \sum_{i=1}^k a_i\}$.

Proof of Theorem 1.1. When $\sum_{i=1}^k a_i \le n-k+2$ or $a_1 \ge (1-\epsilon)n$, Theorem 1.1 follows from Fact 1.3 immediately. When $\sum_{i=1}^k a_i \ge n-k+3$ and $a_1 \le (1-\epsilon)n$, it follows from Theorems 1.4–1.6. ☐

The rest of the paper is organized as follows. In Section 2 we introduce two absorbing lemmas that are needed for the proof of Theorem 1.4: Lemma 2.1 is a simple k-partite version of [23, Fact 2.3]; Lemma 2.2 is derived from a more involved approach by considering the lattice generated by the edges of H. In Sections 3–5, we give the proofs of Theorems 1.4–1.6, respectively. Note that Lemma 2.1, Lemma 3.1, and a portion of Section 4 suffice for the proof of Corollary 1.2 – this was exactly the approach used in [6, 17]. The rest of our proof was carried through with new ideas.

Notation: Given integers $k' \ge k \ge 1$, we write $[k] := \{1, \dots, k\}$ and $[k,k'] := \{k, k+1, \dots, k'\}$. Throughout this paper, we denote by H a k-partite k-graph with the vertex partition $V(H) = V_1 \cup \cdots \cup V_k$. A vertex set S is called *balanced* if it consists of an equal number of vertices from each part of $V(H)$. Given a k-graph H and a set $W \subseteq V(H)$, let $H[W]$ denote the subgraph of H induced on W and $H \setminus W := H[V(H) \setminus W]$.

2. Absorbing Techniques in k-Partite k-Graphs

The main tool in the proof of Theorem 1.4 is the absorbing method. This technique was initiated by Rödl, Ruciński and Szemerédi [21] and has proven to be a powerful tool for finding spanning structures in graphs and hypergraphs. In this section, we prove the absorbing lemmas that will be used in the proof of Theorem 1.4. In fact, we present two different notions of absorbing sets and use them in two different cases.

Let H be a k-partite k-graph. Given a balanced $2k$-set S, an edge $e \in E(H)$ disjoint from S is called *S-absorbing* if there are two disjoint edges $e_1, e_2 \subseteq S \cup \{e\}$ such that $|e_1 \cap S| = k-1$, $|e_1 \cap e| = 1$, $|e_2 \cap S| = 2$, and $|e_2 \cap e| = k-2$. Note that S-absorbing works in the following way: assume that M is a matching such that $S \cap V(M) = \emptyset$ and M contains an S-absorbing edge e, then we can replace e by e_1 and e_2 and get a matching larger than M. Given a crossing k-set S, a set $T \subset V(H) \setminus S$ is called *S-perfect-absorbing* if T is balanced and both $H[T]$ and $H[S\cup T]$ contain perfect matchings. These two definitions work very differently – they are needed for the following two different absorbing lemmas.

Our first absorbing lemma is an analog of [23, Fact 2.3] for k-partite k-graphs.

Lemma 2.1 (Absorbing lemma I)**.** *Let $1/n \ll \alpha \ll \epsilon \ll 1/k$. Suppose H is a k-partite k-graph with parts of size n such that $\delta_{[k]\setminus\{i\}}(H) \ge \epsilon n$ for $i \in [3]$, then there exists a matching M' in H of size at most $\sqrt{\alpha}n$ such that for every balanced $2k$-set S of H, the number of S-absorbing edges in M' is at least αn.*

Our second absorbing lemma deals with the case when only two partite minimum codegrees are large and their sum is not significantly smaller than n.

Lemma 2.2 (Absorbing Lemma II)**.** *Let $1/n \ll \alpha \ll \epsilon \ll 1/t \ll 1/k$. Suppose H is a k-partite k-graph with parts of size n and $a_i := \delta_{[k]\setminus\{i\}}(H)$ for each $i \in [k]$. If $\sum_{i=1}^k a_i \ge (1-\epsilon)n$, $a_1 \ge a_2 \ge \epsilon n$ and $a_j < \epsilon n$ for $j \ge 3$, then one of the following holds.*

(i) *H is $2k^2\epsilon$-D-extremal.*
(ii) *There exists a family $\mathcal{F}'$ of disjoint tk-sets such that $|\mathcal{F}'| \le \sqrt{\alpha}n$, each $F \in \mathcal{F}'$ spans a matching of size t and for every crossing k-set S of H, the number of S-perfect-absorbing sets in $\mathcal{F}'$ is at least αn.*

We first prove the following proposition, which is a standard application of Chernoff's bound. We will apply it in both proofs of Lemmas 2.1 and 2.2 for randomly sampling the absorbing sets.

Proposition 2.3. *Let $1/n \ll \lambda, 1/k, 1/i_0$. Let V be a vertex set with k parts each of size n, and let $\mathcal{F}_1, \dots, \mathcal{F}_t$ be families of balanced i_0k-sets on V such that $|\mathcal{F}_i| \ge \lambda n^{i_0 k}$ for $i \in [t]$ and $t \le n^{2k}$. Then there exists a family $\mathcal{F}' \subseteq \bigcup_{i\in[t]} \mathcal{F}_i$ of disjoint balanced i_0k-sets on V such that $|\mathcal{F}'| \le \lambda n/(4i_0k)$ and $|\mathcal{F}_i \cap \mathcal{F}'| \ge \lambda^2 n/(32 i_0 k)$ for each $i \in [t]$.*

Proof. We build $\mathcal{F}'$ by standard probabilistic arguments. Choose a collection $\mathcal{F}$ of balanced i_0k-sets in H by selecting each balanced i_0k-set on V independently and randomly with probability $p = \epsilon/(2n^{i_0k-1})$, where $\epsilon = \lambda/(4i_0k)$. Since $t \le n^{2k}$, Chernoff's bound implies that, with probability $1-o(1)$, the family $\mathcal{F}$ satisfies the following properties:

$$|\mathcal{F}| \le 2p\binom{n}{i_0}^k \le 2n^{i_0k}p = \epsilon n \quad \text{and} \quad |\mathcal{F}_i \cap \mathcal{F}| \ge \frac{p}{2}\cdot \lambda n^{i_0k} = \frac{1}{4}\lambda\epsilon n \text{ for any } i \in [t].$$

Furthermore, the expected number of intersecting pairs of members in $\mathcal{F}$ is at most

$$p^2 n^{i_0k} \cdot i_0 k \cdot n^{i_0k-1} = \epsilon^2 i_0 k n/4.$$

By Markov's inequality, $\mathcal{F}$ contains at most $\epsilon^2 i_0 k n/2$ intersecting pairs of i_0k-sets with probability at least $1/2$.

Let $\mathcal{F}' \subset \mathcal{F}$ be the subfamily obtained by deleting one i_0k-set from each intersecting pair and removing all i_0k-sets that do not belong to any $\mathcal{F}_i$, $i \in [t]$. Therefore, $|\mathcal{F}'| \le |\mathcal{F}| \le \epsilon n$ and for each $i \in [t]$, we have

$$|\mathcal{F}_i \cap \mathcal{F}'| \ge |\mathcal{F}_i \cap \mathcal{F}| - \frac{1}{2}\epsilon^2 i_0 k n \ge \frac{1}{4}\lambda\epsilon n - \frac{1}{2}\epsilon^2 i_0 k n = \frac{\lambda^2}{32 i_0 k} n$$

and we are done. $\square$

Now we prove our first absorbing lemma.

Proof of Lemma 2.1. We claim that for every balanced $2k$-set S, there are at least $\epsilon^3 n^k/2$ S-absorbing edges. Since there are at most n^{2k} balanced $2k$-sets, the existence of the desired matching follows from Proposition 2.3.

Indeed, assume that $\{w, v\} := S \cap V_3$ and $u \in S \cap V_2$. We obtain S-absorbing edges $e = \{v_1, v_2, \dots, v_k\}$ as follows. First, for each $j \in [4, k]$, we choose arbitrary $v_i \in V_j \setminus S$ – there are $n-2$ choices for each v_j. Having selected $\{v_4, v_5, \dots, v_k\}$, we select a neighbor of $\{u, v, v_4, \dots, v_k\}$ as v_1. Next, we choose a neighbor of S' as v_2, where S' is an arbitrary crossing $(k-1)$-subset of $S \setminus V_2$ that contains w. Finally, we choose a neighbor of $\{v_1, v_2, v_4, \dots, v_k\}$ as v_3. There are at least $\epsilon n - 2$ choices for v_j for $j = 1, 2, 3$. Hence, there are at least

$$(n-2)^{k-3}(\epsilon n - 2)^3 \ge \frac{1}{2}\epsilon^3 n^k > \sqrt{32k\alpha}n^k$$

S-absorbing edges, since n is sufficiently large and $\alpha \ll \epsilon$. Then we get the absorbing matching M' by applying Proposition 2.3 with $\lambda = \sqrt{32k\alpha}$ and $i_0 = 1$. $\square$

The proof of Lemma 2.2 is more involved than that of Lemma 2.1 – we need to apply a *lattice-based absorbing method*, a variant of the absorbing method developed recently by the first author [9]. Roughly speaking, the method provides a vertex partition $\mathcal{P}$ of H (Lemma 2.6) which refines the original k-partition so that we can work on the vectors of $\{0,1\}^{|\mathcal{P}|}$ that represent the edges of H. Using the information obtained from these vectors, we will show that if Lemma 2.2 (ii) does not hold, then H is close to a divisibility barrier based on $\mathcal{P}$. The rest of this section is devoted to the proof of Lemma 2.2, for which we need the following notation and auxiliary results.

The following concepts are introduced by Lo and Markström [16]. Let H be a k-partite k-graph with n vertices in each part. Given $\beta > 0$, $i \in \mathbb{N}, j \in [k]$ and two vertices $u, v \in V_j$, we say that u, v are (β, i)-*reachable in* H if and only if there are at least βn^{ik-1} $(ik-1)$-sets W such that both $H[\{u\} \cup W]$ and $H[\{v\} \cup W]$ contain perfect matchings. In this case W is called a *reachable set* for u, v. A set $X \subseteq V_j$ is (β, i)-*closed in* H if all $u, v \in X$ are (β, i)-reachable in H. Denote by $\tilde{N}_{\beta,i}(v)$ the set of vertices that are (β, i)-reachable to v in H. Clearly, since H is k-partite, for any $j \in [k]$ and $v \in V_j$, $\tilde{N}_{\beta,i}(v) \subseteq V_j$.

We need the following simple fact on k-partite k-graphs.

Fact 2.4. *Let H be a k-partite k-graph with parts of size n. Let $a_1 := \delta_{[k]\setminus\{1\}}(H)$.*

(i) *For any $i \in [2, k]$ and $v \in V_i$, we have $\deg(v) \geq a_1 n^{k-2}$.*
(ii) *If $a_1 \geq (1/3 + \gamma)n$, then for any $i \in [2, k]$, any set of three vertices $u, v, w \in V_i$ contains a pair of vertices which are $(\gamma, 1)$-reachable.*

Proof. To see (i), note that we can obtain an edge containing v by first choosing a $(k-2)$-set $S \in \Pi_{j\neq 1,i} V_j$, and then choosing a neighbor of $\{v\} \cup S$. To see (ii), by (i) and $a_1 \geq (1/3+\gamma)n$, we have $|N(u)|, |N(v)|, |N(w)| \geq (1/3 + \gamma)n^{k-1}$. Also note that $|N(u) \cup N(v) \cup N(w)| \leq n^{k-1}$, then by the inclusion-exclusion principle, we have

$$n^{k-1} \geq |N(u)| + |N(v)| + |N(w)| - |N(u) \cap N(v)| - |N(u) \cap N(w)| - |N(v) \cap N(w)|.$$

So we get $|N(u) \cap N(v)| + |N(u) \cap N(w)| + |N(v) \cap N(w)| \geq 3\gamma n^{k-1}$. Without loss of generality, assume that $|N(u) \cap N(v)| \geq \gamma n^{k-1}$. This implies that u and v are $(\gamma, 1)$-reachable. $\square$

The following proposition reflects the property of $|\tilde{N}_{\epsilon,1}(v)|$.

Proposition 2.5. *Suppose $1/n \ll \epsilon \ll 1/k$ and let H be a k-partite k-graph with n vertices in each part such that $\delta_{[k]\setminus\{1\}}(H), \delta_{[k]\setminus\{2\}}(H) \geq (1/2 - \epsilon)n$. Then for any $j = 1, 2$ and $v \in V_j$, $|\tilde{N}_{\epsilon/3,1}(v)| \geq (1/2 - 2\epsilon)n$. Moreover, for each $j \in [3, k]$, either $|\tilde{N}_{\epsilon/3,1}(v)| \geq \epsilon n$ holds for all vertices $v \in V_j$, or there exists a set $V_j' \subseteq V_j$ of size at most $\epsilon n + 1$ such that $V_j \setminus V_j'$ is $(\epsilon/3, 1)$-closed in H.*

Proof. Fix a vertex $v \in V_j$ for some $j = 1, 2$, note that for any other vertex $u \in V_j$, $u \in \tilde{N}_{\epsilon/3,1}(v)$ if and only if $|N(u) \cap N(v)| \geq \epsilon n^{k-1}/3$. By double counting, we have

$$\sum_{S \in N(v)} (\deg(S) - 1) < |\tilde{N}_{\epsilon/3,1}(v)| \cdot |N(v)| + n \cdot \epsilon n^{k-1}/3.$$

Note that $\sum_{S\in N(v)}(\deg(S)-1) \geq ((1/2-\epsilon)n-1)|N(v)|$. Moreover, by Fact 2.4 (i), $|N(v)| \geq (1/2-\epsilon)n^{k-1} > 2n^{k-1}/5$. Putting these together, we conclude that

$$|\tilde{N}_{\epsilon/3,1}(v)| > \left(\frac{1}{2} - \epsilon\right) n - 1 - \frac{\epsilon n^k/3}{|N(v)|} \geq \left(\frac{1}{2} - \epsilon\right) n - \epsilon n = \left(\frac{1}{2} - 2\epsilon\right) n.$$

Now assume $j \in [3, k]$ and assume that $|\tilde{N}_{\epsilon/3,1}(v)| < \epsilon n$ for some $v \in V_j$. Let $V_j' := \{v\} \cup \tilde{N}_{\epsilon/3,1}(v)$. Thus $|V_j'| \leq \epsilon n + 1$. For any $u, u' \in V_j \setminus V_j'$, since $u \notin \tilde{N}_{\epsilon/3,1}(v)$ and $u' \notin \tilde{N}_{\epsilon/3,1}(v)$, by Fact 2.4 (ii), we conclude that u and u' are $(\epsilon/3, 1)$-reachable. This implies that $V_j \setminus V_j'$ is $(\epsilon/3, 1)$-closed. $\square$

We use the following lemma from [10] to find a partition of each part of H.

Lemma 2.6. [10, Lemma 6.3] *Let $1/m \ll \beta \ll \gamma \ll 1/c, \delta', 1/k$. Suppose that H is an m-vertex k-graph, and a subset $S \subseteq V(H)$ satisfies that $|\tilde{N}_{\gamma,1}(v)| \geq \delta' m$ for any $v \in S$ and every set of $c+1$ vertices in S contains at least two vertices that are $(\gamma, 1)$-reachable. Then we can find a partition $\mathcal{P}_0$ of S into $W_1, \dots, W_d$ with $d \leq \min\{\lfloor 1/\delta' \rfloor, c\}$ such that for any $i \in [d]$, $|W_i| \geq (\delta' - \gamma)m$ and W_i is $(\beta, 2^{c-1})$-closed in H.*

The following useful proposition was proved in [16].

Proposition 2.7. [16, Proposition 2.1] *Let $i \geq 1$, $k \geq 2$ and $1/n \ll \beta \ll \epsilon, \beta, 1/i, 1/k$. Suppose H is a k-graph of order $n \geq n_0$ and there exists a vertex $x \in V(H)$ with $|\tilde{N}_{\beta,i}(x)| \geq \epsilon n$. Then for all $0 < \beta' \leq \beta_0$, $\tilde{N}_{\beta,i}(x) \subseteq \tilde{N}_{\beta',i+1}(x)$.*

Let H be a k-partite k-graph with parts of size n. Suppose $\mathcal{P} = \{W_0, W_1, \dots, W_d\}$ is a partition of $V(H)$ for some integer $d \geq k$ that refines the original k-partition of H. In later applications, W_0 will be so small that we only need to consider the edges not intersecting W_0. The following concepts were introduced by Keevash and Mycroft [11]. The *index vector* of a subset $S \subseteq V(H)$ with respect to $\mathcal{P}$ is the vector

$$\mathbf{i}_{\mathcal{P}}(S) := (|S \cap W_1|, \dots, |S \cap W_d|) \in \mathbb{Z}^d.$$

Given an index vector $\mathbf{v}$, we denote by $\mathbf{v}|_{W_i}$ its value at the coordinate that corresponds to W_i. For $\mu > 0$, define $I^{\mu}_{\mathcal{P}}(H)$ to be the set of all $\mathbf{v} \in \mathbb{Z}^d$ such that H contains at least μn^k edges with index vector $\mathbf{v}$; let $L^{\mu}_{\mathcal{P}}(H)$ denote the lattice in $\mathbb{Z}^d$ generated by $I^{\mu}_{\mathcal{P}}(H)$. For $i \in [d]$, let $\mathbf{u}_{W_i} \in \mathbb{Z}^d$ be the *unit vector* such that $\mathbf{u}_{W_i}|_{W_i} = 1$ and $\mathbf{u}_{W_i}|_{W_j} = 0$ for $j \neq i$.

Lemma 2.8. [7, Lemma 3.4] *Suppose $\beta' \ll \mu, \beta \ll \epsilon \ll 1/i_0, 1/k$ and $1/i' \ll 1/i_0, 1/k$, then the following holds for sufficiently large m. Suppose H is an m-vertex k-graph and $\mathcal{P} = \{W_0, W_1, \dots, W_d\}$ is a partition with $d \leq 2k$ such that $|W_0| \leq \sqrt{\epsilon} m$, $|W_i| \geq \epsilon^2 m$ and W_i is (β, i_0)-closed in H for $i \geq 1$. If $\mathbf{u}_{W_i} - \mathbf{u}_{W_j} \in L^{\mu}_{\mathcal{P}}(H)$, then $W_i \cup W_j$ is (β', i')-closed in H.*

The following lemma shows that if V_1 is closed and $\delta_{[k]\setminus\{1\}}(H) \geq \epsilon n$ then Lemma 2.2 (ii) holds.

Lemma 2.9. *Let $1/n \ll \alpha \ll \beta, \epsilon, 1/i_0, 1/k$. Suppose H is a k-partite k-graph with parts each of size n and $\delta_{[k]\setminus\{1\}}(H) \geq \epsilon n$. If V_1 is (β, i_0)-closed, then there exists a family of disjoint i_0k-sets $\mathcal{F}'$ in H such that $|\mathcal{F}'| \leq \sqrt{\alpha} n$ and each $F \in \mathcal{F}'$ spans a matching of size i_0 and for every crossing k-set S of H, the number of S-perfect-absorbing sets in $\mathcal{F}'$ is at least αn.*

Proof. Fix a crossing k-set $S = \{v_1, v_2, \dots, v_k\}$ such that $v_j \in V_j$, we claim there are at least $\sqrt{32 i_0 k \alpha} n^{i_0 k}$ S-perfect-absorbing i_0k-sets. First of all, we find $v_1' \in V_1 \setminus \{v_1\}$ such that $\{v_1', v_2, \dots, v_k\}$ spans an edge. Since $\deg(S \setminus \{v_1\}) \geq \epsilon n$, there are at least $\epsilon n - 1$ choices for v_1'. Since V_1 is (β, i_0)-closed, there are at least $\beta n^{i_0 k - 1}$ reachable $(i_0 k - 1)$-sets W for v_1 and v_1'. Among them, at least $\beta n^{i_0 k - 1} - (k-1) n^{k-2} \geq \beta n^{i_0 k - 1}/2$ reachable $(i_0 k - 1)$-sets W are disjoint from S. To see that $\{v_1'\} \cup W$ is an S-perfect-absorbing set, note that $H[\{v_1'\} \cup W]$ has a perfect matching by the definition of W, and $H[\{v_1'\} \cup W \cup S]$ has a perfect matching because $\{v_1'\} \cup (S \setminus \{v_1\})$ spans an edge and $H[\{v_1\} \cup W]$ has a perfect matching by the definition of W. In total, we have at least $\epsilon \beta n^{i_0 k}/4 > \sqrt{32 i_0 k \alpha} n^{i_0 k}$ S-perfect-absorbing sets. So we get the family of absorbing sets $\mathcal{F}'$ by applying Proposition 2.3 with $\lambda = \sqrt{32 i_0 k \alpha}$. Note that each $F \in \mathcal{F}'$ is an S-perfect-absorbing set for some crossing k-set S and thus F spans a matching of size i_0. □

Proof of Lemma 2.2. We apply Lemma 2.8 inductively k times, at the jth time with $i = t_{j-1}$ and $i' = t_j$, where $t_0 = 2$. Let $t = t_k$. Pick further constants such that

$$1/n \ll \alpha \ll \beta_k \ll \beta_{k-1} \ll \cdots \ll \beta_1 \ll \mu, \beta \ll \epsilon \ll 1/t_k \ll 1/k.$$

Let H be a k-partite k-graph as given by Lemma 2.2. Suppose that (ii) does not hold. In particular, by Lemma 2.9, we may assume that neither V_1 nor V_2 is (β_k, t_k)-closed in H. By Fact 2.4 (i), we have $\deg(v) \geq a_1 n^{k-2}$ for any $v \notin V_1$, and $\deg(v) \geq a_2 n^{k-2}$ for any $v \in V_1$.

First note that if $a_1 \geq (1/2 + \epsilon)n$, then for any $u, v \in V_2$, we have $|N(u) \cap N(v)| \geq 2\epsilon n^{k-1}$, and thus V_2 is $(2\epsilon, 1)$-closed. By Proposition 2.7, V_2 is (β_k, t_k)-closed, a contradiction.

So we may assume that $a_1 < (1/2 + \epsilon)n$. Thus, we have

$$a_2 \geq \sum_{i=1}^{k} a_i - a_1 - (k-2)\epsilon n \geq (1-\epsilon)n - (1/2 + \epsilon)n - (k-2)\epsilon n = (1/2 - k\epsilon)n,$$

i.e., $(1/2-k\epsilon)n \le a_2 \le a_1 < (1/2+\epsilon)n$. Let $\gamma := (k-1)\epsilon/k$. We apply Proposition 2.5 with $k\epsilon$ in place of ϵ and obtain that, using $\gamma \le \epsilon \le k\epsilon/3$,

(1) for any $i = 1, 2$ and $v \in V_i$, $|\tilde{N}_{\gamma,1}(v)| \ge (1/2 - 2k\epsilon)n$,
(2) for any $i \in [3, k]$, either $|\tilde{N}_{\gamma,1}(v)| \ge k\epsilon n$ for all vertices $v \in V_i$, or there exists a set $V_i' \subseteq V_i$ of size at most $k\epsilon n + 1$ such that $V_i'' = V_i \setminus V_i'$ is $(\gamma, 1)$-closed in H.

Since $a_1, a_2 \ge (1/2-k\epsilon)n \ge (1/3+\gamma)n$, Fact 2.4 (ii) implies that for *any* $i \in [k]$, every set of three vertices of V_i contains two vertices that are $(\gamma, 1)$-reachable in H. Together with (1), it allows us to apply Lemma 2.6 to V_1, V_2 separately with $c = 2$ and $\delta' = 1/(2k) - 2\epsilon$ and partition each of V_1 and V_2 into at most two parts such that each part is $(\beta, 2)$-closed. If V_1 or V_2 is $(\beta, 2)$-closed, then by Proposition 2.7, it is (β_k, t_k)-closed, a contradiction. Thus we assume that each of V_1 and V_2 is partitioned into two parts $V_1 = A_1 \cup B_1$ and $V_2 = A_2 \cup B_2$ such that A_i, B_i are $(\beta, 2)$-closed, and

$$|A_i|, |B_i| \ge (\tfrac{1}{2k} - 2\epsilon - \gamma)kn > (\tfrac{1}{2} - 3k\epsilon)n.$$

Without loss of generality, assume that $|A_1| \le |B_1|$ and $|A_2| \le |B_2|$.

Let I be the set of $i \in [3, k]$ such that $|\tilde{N}_{\gamma,1}(v)| \ge k\epsilon n$ for all vertices $v \in V_i$, and let $I' \subseteq I$ consist of those $i \in I$ such that V_i is not $(\beta, 2)$-closed. We now apply Lemma 2.6 to V_i for $i \in I'$ with $c = 2$ and $\delta' = \epsilon$ and partition V_i into at most two parts such that each part is of size at least $(\epsilon - \gamma)kn = \epsilon n$ and is $(\beta, 2)$-closed. Since V_i, $i \in I'$, is not $(\beta, 2)$-closed, it must be the case that V_i is partitioned into two parts, denoted by A_i and B_i. Let $W_0 = \bigcup_{i\in[3,k]\setminus I} V_i'$ and note that $|W_0| \le (k-2)(k\epsilon n + 1)$. Let $\mathcal{P}_0$ be the partition of $V(H)$ consisting of W_0, A_1, B_1, A_2, B_2 and V_i'' if $i \in [3, k] \setminus I$, V_i if $i \in I \setminus I'$, or A_i, B_i if $i \in I'$. By Proposition 2.7, each part of $\mathcal{P}_0$ except W_0 is $(\beta, 2)$-closed.

For $i \in [k]$ for which A_i and B_i were defined, if $\mathbf{u}_{A_i} - \mathbf{u}_{B_i} \in L^{\mu}_{\mathcal{P}_0}(H)$, then we *merge* A_i and B_i by replacing A_i and B_i with V_i. By Lemma 2.8, $V_i = A_i \cup B_i$ is (β_1, t_1)-closed. We inductively merge A_i, B_i as long as $\mathbf{u}_{A_i} - \mathbf{u}_{B_i} \in L^{\mu}_{\mathcal{P}'}(H)$, where $\mathcal{P}'$ represents the current partition after merging some parts. Since neither V_1 nor V_2 is (β_k, t_k)-closed in H, A_1 and B_1 (also A_2 and B_2) cannot be merged. After at most $k-2$ merges, we obtain a partition $\mathcal{P}$ such that each part except W_0 is (β_k, t_k)-closed (by Proposition 2.7). Write $\mathcal{P} := \{W_0, W_1, \dots, W_d\}$. Let $\tilde{I} \subseteq [k]$ be the set of i such that $A_i, B_i \in \mathcal{P}$ (note that $\mathbf{u}_{A_i} - \mathbf{u}_{B_i} \notin L^{\mu}_{\mathcal{P}}(H)$ for $i \in \tilde{I}$). Write $T := I^{\mu}_{\mathcal{P}}(H) \subseteq \{0,1\}^d$. Given $i, j \in \tilde{I}$ and $\mathbf{w} \in T$, let $\mathbf{w}^i := \mathbf{w} + \mathbf{u}_{A_i} - \mathbf{u}_{B_i} \pmod 2$ and $\mathbf{w}^{i,j} := \mathbf{w} + \mathbf{u}_{A_i} - \mathbf{u}_{B_i} + \mathbf{u}_{A_j} - \mathbf{u}_{B_j} \pmod 2$. We have the following observations.

(†) If $\mathbf{w} \in T$, then $\mathbf{w}^i \notin T$ for $i \in \tilde{I}$.
(‡) If $\mathbf{w} \in T$, then $\mathbf{w}^{i,1} \in T$ for $i \in \tilde{I}$.

Indeed, for (†), if $\mathbf{w} \in T$, then $\mathbf{w}^i \notin T$ because $\mathbf{u}_{A_i} - \mathbf{u}_{B_i} \notin L^{\mu}_{\mathcal{P}}(H)$ for $i \in \tilde{I}$. For (‡), note that $\mathbf{w}^i$ has 1 at k coordinates, which correspond to $W_j \subseteq V_j$, $j \in [k]$ of size at least ϵn (where W_j is V_j or V_j'' or A_j or B_j). The number of the edges in H that contain a crossing $(k-1)$-set in $\prod_{j\in[2,k]} C_j$ is at least $(\epsilon n)^{k-1}a_1$. Since $\mathbf{w}^i \notin T$, there are at most μn^k edges e in H with $\mathbf{i}_{\mathcal{P}}(e) = \mathbf{w}^i$. Consequently, the number of edges e with $\mathbf{i}_{\mathcal{P}}(e) = \mathbf{w}^{i,1}$ is at least $(\epsilon n)^{k-1}a_1 - \mu n^k \ge \mu n^k$, because $\mu \ll \epsilon$ and $a_1 \ge n/3$. Hence $\mathbf{w}^{i,1} \in T$ and this proves (‡).

A vector $\mathbf{v} \in \{0,1\}^d$ is *even* (respectively, *odd*) if there is an even (respectively, odd) number of $i \in \tilde{I}$ such that $\mathbf{v}|_{A_i} = 1$. We claim that all the vectors in T have the same parity. Indeed, assume that there is an even vector $\mathbf{v} \in T$. By (‡), we know that all even vectors are in T. Together with (†), this implies that T contains no odd vector.

Assume that T only contains even vectors (the case when T only contains odd vectors is analogous). Let $A := \bigcup_{i\in\tilde{I}} A_i$ and $B := V \setminus A$. Recall that an edge e of H is even if $|e \cap A|$ is even. Since T only contains even vectors, $E(H \setminus W_0)$ contains at most $2^k\mu n^k$ odd edges. Recall that $(1/2 - 3k\epsilon)n \le |A_1|, |A_2| \le n/2$. In addition, we have shown that $\deg(v) \ge (1/2 - k\epsilon)n^{k-1}$ for all $v \in V(H)$ and thus $|E(H)| \ge (1/2 - k\epsilon)n^k$. Since $\mu \ll \epsilon$ and $|W_0| \le (k-2)(k\epsilon n + 1)$, there are at least

$$|E(H)| - 2^k\mu n^k - |W_0|n^{k-1} \ge (1/2 - k\epsilon)n^k - 2^k\mu n^k - (k-2)(k\epsilon n + 1)n^{k-1} \ge (1/2 - k^2\epsilon)n^k$$

even edges in $E(H)$. Let $|A_1| = n/2 - y$ for some $0 \le y \le 3k\epsilon n$ and assume that the number of odd crossing $(k-1)$-sets in $V \setminus V_1$ is x for some $0 \le x \le n^{k-1}$, then we get

$$\begin{aligned}|E_{even}(A,B)| &= (n/2-y)x + (n/2+y)(n^{k-1}-x)\\ &= n^k/2 + y(n^{k-1}-2x) \le n^k/2 + 3k\epsilon n^k.\end{aligned}$$

Thus we have $|E_{even}(A,B) \setminus E(H)| \le n^k/2 + 3k\epsilon n^k - (1/2 - k^2\epsilon)n^k \le 2k^2\epsilon n^k$. Together with $(1/2 - 3k\epsilon)n \le |A_1|, |A_2| \le n/2$ and $(1/2 - k\epsilon)n \le a_2 \le a_1 < (1/2+\epsilon)n$, and $a_i < \epsilon n$, $3 \le i \le k$, we conclude that H is $2k^2\epsilon$-D-extremal. This completes our proof. $\square$

3. Nonextremal k-partite k-graphs: proof of Theorem 1.4

In this section we first show that every k-partite k-graph H contains an almost perfect matching if $\sum a_i$ is near n and H is not close to H_0. The following lemma is an analog of [6, Lemma 1.7] in k-partite k-graphs. To make it applicable to other problems, we prove it under a weaker assumption which allows a small fraction of crossing $(k-1)$-sets to have small degree.

Lemma 3.1 (Almost perfect matching)**.** *Let $1/n \ll \eta \ll \alpha, \gamma, 1/k$. For $i \in [k]$, let $a_i = a_i(n)$ such that $\sum_{i=1}^k a_i \ge (1-\gamma)n$. Let H be a k-partite k-graph with parts of size n which is not 2γ-S-extremal. Suppose for each $i \in [k]$, there are at most ηn^{k-1} crossing $(k-1)$-sets S such that $S \cap V_i = \emptyset$ and $\deg(S) < a_i$. Then H contains a matching that covers all but at most αn vertices in each vertex class.*

Proof. Let M be a maximum matching in H and assume $m = |M|$. Let $V_i' = V_i \cap V(M)$ and $U_i = V_i \setminus V(M)$. Suppose to the contrary, that $|U_1| = \cdots = |U_k| > \alpha n$.

Let $t = \lceil k(k-1)/\gamma \rceil$. We find a family $\mathcal{A}$ of disjoint crossing $(k-1)$-subsets $A_1, \dots, A_{kt}$ of $V \setminus V(M)$ such that $A_j \cap V_i = \emptyset$ and $\deg(A_j) \ge a_i$ whenever $j \equiv i \bmod k$. This can be done greedily because when selecting A_j, the crossing $(k-1)$-sets that cannot be picked are either those that intersect the ones that have been picked, or those with low degree, whose number is at most

$$k(k-1)tn^{k-2} + \eta n^{k-1} < (\alpha n)^{k-1} < \prod_{\ell \in [k]\setminus\{i\}} |U_\ell|,$$

because $1/n \ll \eta \ll \alpha$. Note that the neighbors of A_j are in V_i' with $j \equiv i \bmod k$ by the maximality of M.

For $i \in [k]$, let D_i be the set of the vertices of V_i' that have at least k neighbors in $\mathcal{A}$ and let $D = \bigcup D_i$. We claim that $|e \cap D| \le 1$ for each $e \in M$. Indeed, otherwise assume that $x, y \in e \cap D$ and pick A_i, A_j for some $i, j \in [kt]$ such that $\{x\} \cup A_i, \{y\} \cup A_j \in E(H)$. We obtain a matching of size $m+1$ by deleting e and adding $\{x\} \cup A_i$ as well as $\{y\} \cup A_j$ in M, contradicting the maximality of M.

Next we show that $|D_i| \ge a_i - \gamma n/k$ for each $i \in [k]$. Since $N(A_j) \cap V_i' = \emptyset$ for $j \not\equiv i \bmod k$, we get

$$t \cdot a_i \le \sum_{j \equiv i \bmod k} \deg(A_j) \le |D_i|t + n(k-1).$$

Since $t \ge k(k-1)/\gamma$, it follows that

$$|D_i| \ge a_i - \frac{n(k-1)}{t} \ge a_i - \frac{\gamma n}{k}.$$

This implies that $|D| = \sum_{i=1}^k |D_i| \ge \sum_{i=1}^k a_i - \gamma n$. Since every edge of M contains at most one vertex of D, we have $|D| \le |M| < n$ and consequently, $\sum_{i=1}^k a_i \le n + \gamma n = (1+\gamma)n$.

Define $M' := \{e \in M : e \cap D \neq \emptyset\}$. Then for each $i \in [k]$, we have

$$|(V(M') \setminus D) \cap V_i| = \sum_{j \neq i} |D_j| \ge \sum_{j \in [k]} (a_j - \tfrac{\gamma n}{k}) - a_i \ge n - a_i - 2\gamma n.$$

Since H is not 2γ-S-extremal, $H[V(M') \setminus D]$ contains at least one edge, denoted by e_0. Note that $e_0 \notin M$ because each edge of M' contains exactly one vertex of D and $e_0 \subset V(M') \setminus D$. Assume that e_0 intersects $e_1, \dots, e_p$ in M for some $2 \le p \le k$. Suppose $\{v_j\} := e_j \cap D$. Note that $v_j \notin e_0$ for all $j \in [p]$. Since each v_j has at least k neighbors in $\mathcal{A}$, we can greedily pick $A_{\ell_1}, \dots, A_{\ell_p} \in \mathcal{A}$ such that $\{v_j\} \cup A_{\ell_j} \in E(H)$ for all $j \in [p]$. Let M'' be the matching obtained from M after replacing $e_1, \dots, e_p$ by e_0 and $\{v_j\} \cup A_{\ell_j}$ for $j \in [p]$. Thus, M'' has $m+1$ edges, contradicting the choice of M. $\square$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $1/n \ll \eta \ll \alpha \ll \gamma \ll \epsilon \ll 1/t \ll 1/k$. Suppose both (ii) and (iii) fail and we will show that (i) holds.

First assume that $a_1 \ge a_2 \ge a_3 \ge \epsilon n$. We first apply Lemma 2.1 to H and find a matching M' of size at most $\sqrt{\alpha}n$ such that for every balanced $2k$-set $S \subset V(H)$, the number of S-absorbing edges in M' is at least αn. Let $H' := H \setminus V(M')$, $n' := |V(H') \cap V_i| \ge (1-\sqrt{\alpha})n$ and $a_i' := \delta_{[k]\setminus\{i\}}(H')$. Note that $\sum_{i=1}^k a_i' \ge \sum_{i=1}^k a_i - k\sqrt{\alpha}n \ge (1-2\gamma/5)n'$. Assume for a moment that H' is $(4\gamma/5)$-S-extremal, i.e., $\sum_{i=1}^k a_i' \le n' + (4\gamma/5)n'$ and $V(H')$ contains an independent set C such that $|C \cap (V_i \cap V(H'))| \ge n' - a_i' - 4\gamma n'/5$ for each $i \in [k]$. Then as $\alpha \ll \gamma$, it follows that $\sum_{i=1}^k a_i \le \sum_{i=1}^k a_i' + k\sqrt{\alpha}n \le n' + (4\gamma/5)n' + k\sqrt{\alpha}n \le n + \gamma n$ and

$$n' - a_i' - 4\gamma n'/5 \ge (1-\sqrt{\alpha})n - a_i - 4\gamma n/5 \ge n - a_i - \gamma n,$$

This means that H is γ-S-extremal, a contradiction. Thus, H' is not $(4\gamma/5)$-S-extremal. By applying Lemma 3.1 to H' with parameters $2\gamma/5$, α and η, we obtain a matching M'' in H' that covers all but at most αn vertices in each vertex class.

Since there are at least αn S-absorbing edges in M' for every balanced $2k$-set $S \subset V(H)$, we can repeatedly absorb the leftover vertices until there is one vertex left in each class. Denote by $\tilde{M}$ the matching obtained after absorbing the leftover vertices into M'. Therefore $\tilde{M} \cup M''$ is the required matching of size $n-1$ in H.

Secondly assume that $a_1 \ge a_2 \ge \epsilon n$ and $a_i < \epsilon n$ for $i \in [3,k]$. Since (iii) does not hold, by applying Lemma 2.2, there exists a family of disjoint absorbing tk-sets $\mathcal{F}'$ of size $|\mathcal{F}'| \le \sqrt{\alpha}n$ such that each $F \in \mathcal{F}'$ spans a matching of size t and for every crossing k-set S of H, the number of S-perfect-absorbing sets in $\mathcal{F}'$ is at least αn. Let $H' := H \setminus V(M')$ and $n' := |V(H') \cap V_i| \ge (1-t\sqrt{\alpha})n$ and $a_i' := \delta_{[k]\setminus\{i\}}(H')$. Note that $\sum_{i=1}^k a_i' \ge \sum_{i=1}^k a_i - kt\sqrt{\alpha}n \ge (1-2\gamma/5)n'$ as $\alpha \ll \epsilon, 1/t$. As before, we may assume that H' is not $(4\gamma/5)$-S-extremal. By applying Lemma 3.1 to H' with parameters $2\gamma/5$, α and η, we obtain a matching M'' in H' that covers all but at most αn vertices in each vertex class. Let U be the set of leftover vertices. Since any crossing k-subset S of U has at least αn S-perfect-absorbing tk-sets in $\mathcal{F}'$, we can greedily absorb all the leftover vertices into $\mathcal{F}$. Denote by $\tilde{M}$ the resulting matching that covers $V(\mathcal{F}') \cup U$. We obtain a perfect matching $\tilde{M} \cup M''$ of H. $\square$

4. Proof of Theorem 1.5

We prove Theorem 1.5 in this section. Following the approach in [6], we use the following weaker version of a result by Pikhurko [19]. Let H be a k-partite k-graph with parts $V_1, \dots, V_k$. Given $L \subseteq [k]$, recall that

$$\delta_L(H) = \min\left\{\deg(S) : S \in \prod_{i \in L} V_i\right\}.$$

Lemma 4.1. [19, Theorem 3] *Given $k \ge 2$ and $L \subseteq [k]$, let m be sufficiently large. Let H be a k-partite k-graph with parts $V_1, \dots, V_k$ of size m. If*

$$\delta_L(H)m^{|L|} + \delta_{[k]\setminus L}(H)m^{k-|L|} \ge \frac{3}{2}m^k,$$

then H contains a perfect matching.

Proof of Theorem 1.5. Let $\alpha = \sqrt{\gamma}$. Assume that H is not 3ϵ-D-extremal. Our goal is to find a matching in H of size at least $\min\{n-1, \sum_{i=1}^k a_i\}$. Assume that H is γ-S-extremal, namely, $\sum_{i=1}^k a_i \le n + \gamma n$ and there is an independent set $C \subseteq V(H)$ such that $|C \cap V_i| \ge n - a_i - \gamma n$ for each $i \in [k]$.

We may assume that $\sum_{i=1}^k a_i \ge n-k+3$, as otherwise we are done by Fact 1.3. So we have

$$n - k + 3 \le \sum_{i=1}^k a_i \le n + \gamma n. \tag{4.1}$$

For each $i \in [k]$, let $C_i := C \cap V_i$. We know that $|C_i| \ge n - a_i - \gamma n \ge (\epsilon - \gamma)n$ from the assumption that $a_i \le (1-\epsilon)n$. We partition each $V_i \setminus C_i$ into $A_i \cup B_i$ such that

$$A_i := \left\{x \in V_i \setminus C_i : \deg(x, C) \ge (1-\alpha)\prod_{j \ne i}|C_j|\right\}, \tag{4.2}$$

and $B_i := V_i \setminus (A_i \cup C_i)$. Moreover, let $A := \bigcup_{1\le i\le k} A_i$ and $B := \bigcup_{1\le i\le k} B_i$.

Claim 4.2. *For $i \in [k]$, we have*
(1) $a_i \le |A_i| + |B_i| \le a_i + \gamma n$,
(2) $|B_i| \le \alpha n$, *and*
(3) $a_i - \alpha n \le |A_i| \le a_i + \gamma n$.

Proof. For $i \in [k]$, since $|C_i| \ge n - a_i - \gamma n$, we have $|A_i| + |B_i| \le a_i + \gamma n$. For any crossing $(k-1)$-set $S \subset C \setminus V_i$, we have $N(S) \subseteq A_i \cup B_i$. By the codegree condition, we have $|A_i| + |B_i| \ge a_i$.

Let E_i denote the set of the edges that consist of a $(k-1)$-set in $\prod_{j\ne i} C_i$ and one vertex in $A_i \cup B_i$. By the definition of A_i, we have

$$a_i \prod_{j\ne i} |C_j| \le |E_i| \le |B_i|(1-\alpha)\prod_{j\ne i}|C_j| + |A_i|\prod_{j\ne i}|C_j|,$$

which implies $a_i \le |A_i| + |B_i| - \alpha|B_i|$. It follows that $\alpha|B_i| \le |A_i| + |B_i| - a_i \le \gamma n$ by Part (1). Since $\alpha = \sqrt{\gamma}$, it follows that $|B_i| \le \alpha n$.

Part (3) follows from Parts (1) and (2) immediately. □

Our procedure towards the desired matching consists of three steps. First, we remove a matching that covers all the vertices of B. Secondly, we remove another matching in order to have $|C_i''| - \sum_{j\ne i} |A_j''| \le \max\{1, n - \sum_{i=1}^k a_i\}$ for all $i \in [k]$, where C_i'' and A_i'' denote the set of the remaining vertices in C_i and A_i, respectively. Finally, we apply Lemma 4.1 to get a matching that covers all but at most $\max\{1, n - \sum_{i=1}^k a_i\}$ vertices in each V_i.

Step 1. Cover the vertices of B.

For $i \in [k]$, define $t_i := \max\{0, a_i - |A_i|\}$. By Claim 4.2 (1), we have $|B_i| \ge a_i - |A_i|$. Together with the definition of t_i and Claim 4.2 (2), we have

$$t_i \le |B_i| \le \alpha n \quad \text{and} \quad t_i + |A_i| \ge a_i. \tag{4.3}$$

First we build a matching M_1^i of size t_i for each $i \in [k]$ and let M_1 be the union of them. If $t_i = 0$, then $M_1^i = \emptyset$. Otherwise, since $a_i = \delta_{[k]\setminus\{i\}}(H)$ and C is independent, every $(k-1)$-set in $\prod_{j\ne i} C_j$ has at least $a_i - |A_i| = t_i$ neighbors in B_i. We greedily pick t_i disjoint edges each of which consists of a $(k-1)$-set in $\prod_{j\ne i} C_j$ and one vertex in B_i.

Next for each i, we greedily build a matching M_2^i that covers all the remaining vertices in B_i and let M_2 be the union of them. Indeed, for each of the remaining vertices $v \in B_i$ with $i \ne 1$, we pick one uncovered $(k-2)$-set S' in $\prod_{j\ne i,1} C_j$, and one uncovered vertex in $N(\{v\}\cup S') \subseteq V_1$. For each of the remaining vertices in $v \in B_1$, we pick one uncovered $(k-2)$-set S' in $\prod_{j\ne 1,2} C_j$, and one uncovered vertex in $N(\{v\}\cup S') \subseteq V_2$. Since the number of vertices in V_i covered by the existing matchings is at most $|M_1 \cup M_2| \le |B| \le k\alpha n < \epsilon n \le a_2 \le a_1$, we can always find an uncovered vertex from $N(\{v\}\cup S')$.

For $i \in [k]$, let

$$A_i' := A_i \setminus V(M_1 \cup M_2),\ C_i' := C_i \setminus V(M_1 \cup M_2) \text{ and } V_i' := V_i \setminus V(M_1 \cup M_2).$$

Step 2. Adjust the sizes of A_i' and C_i'.

In this step, we will build a small matching M_3 in order to adjust the sizes of A_i' and C_i'.

Claim 4.3. *There exists a matching M_3 of size at most $2k\gamma n$ in $H[\bigcup_{i=1}^k V_i']$ so that $|C_i' \setminus V(M_3)| - \sum_{j\ne i} |A_j' \setminus V(M_3)| = r$ for some integer $0 \le r \le \max\{1, n - \sum_{i=1}^k a_i\}$.*

Proof. Let $n' := |V_i'| = |A_i'| + |C_i'|$. Let $s_0 := |C_i'| - \sum_{j\ne i} |A_j'| = n' - \sum_{j=1}^k |A_j'|$, which is independent of i. We claim that $-2k\gamma n \le s_0 \le n - \sum_{i=1}^k a_i$. Indeed,

$$s_0 \ge (n - (|M_1| + |M_2|)) - |A| \ge n - |B| - |A| \overset{\text{Claim 4.2}}{\ge} n - \sum_{i=1}^k (a_i + \gamma n) \overset{(4.1)}{\ge} -2k\gamma n.$$

On the other hand, since $V(M_1) \cap A = \emptyset$ and $|M_1| = \sum_{j=1}^k t_j$, we have

$$s_0 \le n - (|M_1| + |M_2|) - \left(\sum_{j=1}^k |A_j| - |M_2| \right) = n - \sum_{j=1}^k (t_j + |A_j|) \overset{(4.3)}{\le} n - \sum_{j=1}^k a_j.$$

If $s_0 \ge 0$, then set $M_3 = \emptyset$ and we are done. Otherwise, we build M_3 by adding edges that contain two or three vertices of A one by one until $s \in \{0,1\}$, where $s := (n' - |M_3|) - \sum |A'_j \setminus V(M_3)|$. This will be done in the next few paragraphs. Note that since $s_0 \ge -2k\gamma n$ and adding an edge to M_3 increases s by one or two, we will have $|M_3| \le 2k\gamma n$.

Now we show how to build M_3. First assume that $a_3 \ge 2k\alpha n$. In this case we greedily choose the edges of M_3 until $s \in \{0,1\}$ by picking two uncovered vertices, one from A'_2 and one from A'_3, an uncovered $(k-3)$-set in $\prod_{j\in[4,k]} C'_j$, and one uncovered vertex in V'_1 by the degree condition. To see why we can find these edges, first, we can always pick two uncovered vertices in $A'_2 \cup A'_3$ because by Claim 4.2 (3),

$$|A'_i| \ge |A_i| - |M_2| \ge a_i - \alpha n - k\alpha n \ge 2k\gamma n \tag{4.4}$$

for $i = 2, 3$. Secondly, we can find an uncovered $(k-3)$-set in $\prod_{j\in[4,k]} C'_j$ because

$$|C'_j| \ge |C_j| - |M_1 \cup M_2| \ge \epsilon n - \gamma n - k\alpha n \ge 2k\gamma n. \tag{4.5}$$

Thirdly, we can find the desired vertex in V_1 because the number of covered vertices in V_1 is at most $|B| + 2k\gamma n \le 2k\alpha n < a_1$.

Next assume that $|A_1| \ge (1/2 + \epsilon)n$. In this case we greedily choose the edges of M_3 until $s \in \{0,1\}$ by picking an uncovered vertex in A'_2, an uncovered $(k-2)$-set in $\prod_{j\in[3,k]} C'_j$, and by the degree condition, one uncovered vertex in A'_1. To see why we can find these edges, first, we can pick an uncovered $(k-1)$-set $S \in A'_2 \times \prod_{i\in[3,k]} C'_i$ because of (4.4) and (4.5). Secondly, note that $a_1 \ge |A_1| - \gamma n \ge (1/2 + \epsilon - \gamma)n$ and

$$|A'_1| \ge |A_1| - |M_2| \ge (\tfrac{1}{2} + \epsilon)n - k\alpha n = (\tfrac{1}{2} + \epsilon - k\alpha)n.$$

Thus, S has at least $a_1 - (n - |A'_1|) \ge 2k\gamma n$ neighbors in A'_1 so we can find an uncovered neighbor of S.

Now we assume that $|A_1| < (1/2+\epsilon)n$ and $a_3 < 2k\alpha n \le \epsilon n$. In this case we show that (ii) holds, i.e., H is 3ϵ-D-extremal. First, $a_1 \le |A_1| + \alpha n < (1/2 + \epsilon + \alpha)n$. Since $a_i \le a_3$ for $i \in [3,k]$ and $\sum_{i=1}^k a_i \ge n - k + 3$, we have

$$a_2 \ge \sum_{i=1}^k a_i - a_1 - (k-2)a_3 \ge n - k + 3 - (\tfrac{1}{2} + \epsilon + \alpha)n - 2k(k-2)\alpha n \ge (\tfrac{1}{2} - 2\epsilon)n,$$

i.e., $(1/2 - 2\epsilon)n \le a_2 \le a_1 \le (1/2 + 2\epsilon)n$. By Claim 4.2 (3), $|A_2| \le (1/2 + 2\epsilon)n + \gamma n \le (1/2 + 3\epsilon)n$ and $|A_i| \le a_i + \gamma n \le 3k\alpha n$ for $i \in [3,k]$. The lower bounds on a_1, a_2 implies that $|A_1|, |A_2| \ge (1/2 - 2\epsilon)n - \alpha n \ge (1/2 - 3\epsilon)n$. Finally, let x be the number of crossing k-sets in $V(H)$ that intersect A_i for some $i \in [3,k]$, then $x \le (k-2)3k\alpha n \cdot n^{k-1} \le 3k^2\alpha n^k$. Let y_1 be the number of non-edges in $H[A_1, B_2 \cup C_2, \dots, B_k \cup C_k]$ and let y_2 be the number of non-edges in $H[B_1 \cup C_1, A_2, B_3 \cup C_3, \dots, B_k \cup C_k]$. By the definition of A and $|B_i| \le \alpha n$, for $1 \le i \le k$ we have

$$y_i \le |A_i| \cdot \alpha \prod_{j\ne i} |C_j| + \sum_{j \ne i} |B_j| \cdot n^{k-1} \le \alpha n^k + (k-1)\alpha n^k = k\alpha n^k.$$

Note that $|E_{odd}(A, B\cup C) \setminus E(H)| \le x + y_1 + y_2 \le 3k^2\alpha n^k + 2 \cdot k\alpha n^k \le \epsilon n^k$. So (ii) holds, a contradiction. □

Step 3. Cover the remaining vertices.

Let M_3 and r be as in Claim 4.3. For each $i \in [k]$, let

$$A''_i := A'_i \setminus V(M_3),\ C''_i := C'_i \setminus V(M_3) \text{ and } V''_i := V'_i \setminus V(M_3).$$

By the definitions of M_1, M_2, M_3, we have $|M_1 \cup M_2 \cup M_3| \le k\alpha n + 2k\gamma n \le (k+1)\alpha n$. By Claim 4.2 (3), for each $i \in [k]$, we have

$$|A''_i| \ge |A_i| - |M_1 \cup M_2 \cup M_3| \ge (a_i - \alpha n) - (k+1)\alpha n \ge a_i - 2k\alpha n.$$

Recall that $a_1 \ge a_2 \ge \epsilon n$, by $\gamma \ll \epsilon$, we have

$$|A''_1|, |A''_2| \ge a_2 - 2k\alpha n \ge \epsilon n/2. \tag{4.6}$$

By Claim 4.3, we have

$$0 \le r = |C_i''| - \sum_{j\neq i} |A_j''| \le \max\left\{1, n - \sum_{i=1}^{k} a_i\right\}. \tag{4.7}$$

For $i \in [k]$, since $a_i \le (1-\epsilon)n$, we get that $|C_i| \ge \epsilon n - \gamma n \ge 2(k+1)\alpha n$. Thus,

$$|C_i''| \ge |C_i| - |M_1 \cup M_2 \cup M_3| \ge |C_i| - (k+1)\alpha n \ge |C_i|/2. \tag{4.8}$$

Now we greedily cover the vertices of $A_3'', \dots, A_k''$ with disjoint edges of H. Indeed, for every $3 \le i \le k$ and every vertex $v \in A_i''$, we find a neighbor of v from $\bigcup_{j\neq i} C_i''$. By (4.2) and (4.8), at most

$$\alpha \prod_{j\neq i} |C_j| \le 2^{k-1}\alpha \prod_{j\neq i} |C_j''|$$

crossing $(k-1)$-sets in $\prod_{j\neq i} C_j''$ are *not* neighbors of v. Since $|C_i''| = \sum_{j\neq i} |A_j''| + r$, at least $\min\{|A_1''| + r, |A_2''| + r\} \ge \epsilon n/2$ vertices of C_i'' remain at the end of the greedy process. The greedily algorithm works because $(\epsilon n/2)^{k-1} \ge 2^{k-1}\alpha n^{k-1} > 2^{k-1}\alpha \prod_{j\neq i} |C_j''|$.

Let M_4 be the resulting matching in this step. Let $m_i := |A_i''|$ for all $i = 1, 2$. Note that there are $m_2 + r$ and $m_1 + r$ remaining vertices in C_1'', C_2'', respectively, and $m_1 + m_2 + r$ remaining vertices in C_i'' for $i \ge 3$. Our goal is to cover all the remaining vertices of H. For $i = 1, 2$, let C_1^2 be a set of m_2 vertices in $C_1'' \setminus V(M_4)$ and let C_2^1 be a set of m_1 vertices in $C_2'' \setminus V(M_4)$; for $i \in [3, k]$, we can partition all but r vertices of $C_i'' \setminus V(M_4)$ into C_i^1 of size m_1 and C_i^2 of size m_2. Therefore, we get k-partite k-graphs $H_i := H[A_i'' \cup \bigcup_{\ell\neq i} C_\ell^i]$ for $i = 1, 2$. Below we verify the assumptions of Lemma 4.1 for H_i.

First, for $i \in [2]$ and any $(k-1)$-set $S \in \prod_{\ell\neq i} C_\ell^i$, the number of its non-neighbors in $A_i \cup B_i$ is at most

$$|A_i| + |B_i| - a_i \overset{\text{Claim 4.2}}{\le} \gamma n \overset{(4.6)}{\le} \gamma \cdot \frac{2}{\epsilon} m_i \le \alpha m_i,$$

as $\gamma \ll \epsilon$ and $\alpha = \sqrt{\gamma}$. So we have

$$\delta_{[k]\setminus\{i\}}(H_i) \ge m_i - \alpha m_i = (1-\alpha)m_i.$$

Next, for any $v \in A_i''$, by (4.2) the number of its non-neighbors in $\prod_{\ell\neq i} C_\ell^i$ is at most

$$\alpha \prod_{\ell\neq i} |C_\ell| < \alpha n^{k-1} \overset{(4.6)}{\le} \alpha \left(\frac{2}{\epsilon} m_i\right)^{k-1} \le \sqrt{\alpha} m_i^{k-1},$$

which implies that $\delta_{\{i\}}(H_i) \ge (1-\sqrt{\alpha})m_i^{k-1}$. Thus, we have

$$\delta_{\{i\}}(H_i)m_i + \delta_{[k]\setminus\{i\}}(H_i)m_i^{k-1} \ge (1-\sqrt{\alpha})m_i^{k-1}m_i + (1-\alpha)m_i m_i^{k-1} > \frac{3}{2}m_i^k,$$

since γ is small enough. By Lemma 4.1, we find a perfect matching M_5^i in H_i for each $i \in [2]$. Let $M_5 := M_5^1 \cup M_5^2$, then $M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5$ is a matching in H of size at least $n - r$. If $r \le 1$, then we obtain a matching of size at least $n-1$. Otherwise, since $0 < r \le n - \sum_{i=1}^k a_i$, we get a matching of size at least $\sum_{i=1}^k a_i$. □

5. Proof of Theorem 1.6

We call a binary vector $\mathbf{v} \in \{0,1\}^k$ *even* (otherwise *odd*) if it contains an even number of coordinates that have value 1. Let EV_k denote the set of all even vectors in $\{0,1\}^k$. Note that $|EV_k| = 2^{k-1}$. Let $H = (V, E)$ be a k-partite k-graph with parts $V_1, \dots, V_k$. Suppose V also has a partition $A \cup B$, and let $A_i := A \cap V_i$ and $B_i := B \cap V_i$ for $i \in [k]$. Recall that a set $S \subseteq V$ is *even* (or *odd*) if $|S \cap A|$ is even (or odd) and $E_{even}(A, B)$ consists all *crossing* even k-subsets of V. Given a vector $\mathbf{v} \in \{0,1\}^k$, we write $V^{\mathbf{v}} = V_1^{\mathbf{v}} \cup \cdots \cup V_k^{\mathbf{v}}$, where $V_i^{\mathbf{v}} := A_i$ if $\mathbf{v}|_{V_i} = 1$ and $V_i^{\mathbf{v}} := B_i$ otherwise. Let $H(\mathbf{v}) := H[V^{\mathbf{v}}]$. For any crossing k-set $S \in V^{\mathbf{v}}$, we say that $\mathbf{v}$ is the *location vector* of S. For $v \in V$, we define $\overline{\deg}_H(v) := \prod_{j\neq i} |V_j| - \deg_H(v)$, which is the degree of v in the complement of H under the same k-partition. Let $\overline{\delta_1}(H) := \max_{v\in V(H)} \overline{\deg}_H(v)$.

The following theorem is a key step in the proof of Theorem 1.6.

Theorem 5.1. *Suppose $1/n \ll \eta \ll \epsilon_0, 1/k$ and n is an even integer. Let H be a k-partite k-graph with parts $V_1, \dots, V_k$ of size n. Suppose $A \cup B$ is a partition of $V(H)$ with $A_i := A \cap V_i$ and $B_i := B \cap V_i$ such that*

(i) $|A_1| = |A_2| = n/2$,
(ii) $|A_i| = 0$ *or* $\epsilon_0 n \le |A_i| \le (1-\epsilon_0)n$ *for* $i \ge 3$,
(iii) *for any even vector* $\mathbf{v}$, $\overline{\delta_1}(H(\mathbf{v})) \le \eta n^{k-1}$.

Then H contains a matching of size $n-1$. Furthermore, if $|A|$ is even, then H contains a perfect matching.

To prove Theorem 5.1, we need the following simple result.

Lemma 5.2. *Given a set V of kn vertices for some even integer n, let $V_1 \cup \dots \cup V_k$ and $A \cup B$ be two partitions of V such that $|V_1| = \dots = |V_k| = n$ and $|A_1| = |A_2| = n/2$, where $A_i := A \cap V_i$ and $B_i := B \cap V_i$. Let $H = (V, E_{even}(A, B))$. If $|A|$ is odd, then H contains a matching of size $n-1$; if $|A|$ is even, then H contains a perfect matching.*

Proof. We first prove the case when $|A|$ is even by induction on n. The base case $n = 2$ is simple: we divide the $2k-2$ vertices in $\bigcup_{i \ge 2} V_i$ arbitrarily into two $(k-1)$-sets and add the vertices of V_1 to make both sets even (these two k-sets have the same parity because $|A|$ is even). For the induction step, assume $n \ge 4$ (as n is even). By picking two vertices in V_i, $i \ge 3$, with the same parity, we find two disjoint crossing $(k-2)$-sets in $\bigcup_{i \ge 3} V_i$ with the same parity. We next extend them to two even k-sets by adding four vertices, one from each of A_1, A_2, B_1, B_2. Since both k-sets are even, after deleting them, we can apply the inductive hypothesis.

For the case when $|A|$ is odd, we apply the previous case after moving one vertex from $\bigcup_{i \in [3,k]} A_i$ to B (note that $\sum_{i \in [3,k]} |A_i|$ is odd because $|A_1| + |A_2| = n$ is even). Since exactly one edge has the 'wrong' parity, we obtain a matching of H of size $n-1$. $\square$

We also use the following result of Daykin and Häggkvist [4] while proving Theorem 5.1.

Theorem 5.3. [4] *Let $1/n \ll 1/k$. If H is a k-partite k-graph with parts of size n such that $\delta_1(H) \ge (1 - 1/k)(n^{k-1} - 1)$, then H contains a perfect matching.*

Proof of Theorem 5.1. We first note that for any $\mathbf{v} \in EV_k$ and for arbitrary subsets $U_i \subseteq V_i^{\mathbf{v}}$, $i \in [k]$, such that $|U_i| \ge \eta^{1/(2k)} n$, (iii) implies that

$$\overline{\delta_1}(H[U_1, \dots, U_k]) \le \eta n^{k-1} \le \eta^{1/2} \prod_{2 \le i \le k} |U_i|. \tag{5.1}$$

We now apply Lemma 5.2 to $H' := (V, E_{even}(A, B))$ and conclude that H' contains a matching M of size at least $n-1$; moreover, M is perfect if $|A|$ is even. Let $S := V \setminus V(M)$. For each $\mathbf{v} \in EV_k$, let $m_{\mathbf{v}}$ be the number of edges in M with location vector $\mathbf{v}$. Then $\sum_{\mathbf{v} \in EV_k} m_{\mathbf{v}} = |M|$ as all the edges in M are even.

It suffices to build a matching of H that consists of $m_{\mathbf{v}}$ edges with location vector $\mathbf{v}$ for each $\mathbf{v} \in EV_k$. Let us partition EV_k into $\mathcal{V}_1 \cup \mathcal{V}_2$ such that $\mathcal{V}_1$ consists of all $\mathbf{v}$ with $m_{\mathbf{v}} < \eta^{1/(2k)} n$. For each $\mathbf{v} \in \mathcal{V}_1$, we greedily find a matching of size $m_{\mathbf{v}}$ in $H_{\mathbf{v}}$. To see why this is possible, note that in total at most $2^{k-1} \eta^{1/(2k)} n \le \epsilon_0^2 n$ edges of M have their location vectors in $\mathcal{V}_1$. Consequently the number of the crossing $(k-1)$-sets in $V_2^{\mathbf{v}} \cup \dots \cup V_k^{\mathbf{v}}$ that intersect these edges is at most

$$\epsilon_0^2 n \sum_{2 \le i \le k} \prod_{2 \le j \le k, j \ne i} |V_j^{\mathbf{v}}| \le (k-1)\epsilon_0 \prod_{2 \le i \le k} |V_i^{\mathbf{v}}|$$

because $|V_i^{\mathbf{v}}| \ge \epsilon_0 n$ for $2 \le i \le k$. By (5.1), for any $v \in V_1^{\mathbf{v}}$, we have $\deg_{H(\mathbf{v})}(v) \ge (1 - \eta^{1/2}) \prod_{2 \le i \le k} |V_i^{\mathbf{v}}| > (k-1)\epsilon_0 \prod_{2 \le i \le k} |V_i^{\mathbf{v}}|$ – this guarantees the existence of the desired matchings for all $\mathbf{v} \in \mathcal{V}_1$.

Next we arbitrarily divide the remaining vertices of $V \setminus S$ into balanced vertex partitions $U_{\mathbf{v}} = U_1^{\mathbf{v}} \cup \dots \cup U_k^{\mathbf{v}}$, $\mathbf{v} \in \mathcal{V}_2$, such that $U_i^{\mathbf{v}} \subseteq V_i^{\mathbf{v}}$ and $|U_1^{\mathbf{v}}| = \dots = |U_k^{\mathbf{v}}| = m_{\mathbf{v}}$ – this is possible because $\sum_{\mathbf{v} \in EV_k} m_{\mathbf{v}} = |M|$. By (5.1), we know that $\delta_1(H[U_{\mathbf{v}}]) \ge (1 - \eta^{1/2}) m_{\mathbf{v}}^{k-1} \ge (1 - 1/k) m_{\mathbf{v}}^{k-1}$ as η is small enough. We thus apply Theorem 5.3 to each $H[U_{\mathbf{v}}]$ and get a perfect matching of $H[U_{\mathbf{v}}]$. Putting all the matchings that we obtained together gives a matching of size $|M|$ in H. $\square$

Proof of Theorem 1.6. Pick a new constant ϵ_0 such that $\epsilon \ll \epsilon_0 \ll 1/k$. We assume that $\sum_{i=1}^k a_i \ge n-k+3$ – otherwise we are done by Fact 1.3. Moreover, suppose H has a vertex bipartition $A \cup B = V_1 \cup \cdots \cup V_k$ such that

(†) $(1/2-\epsilon)n \le a_1, a_2, |A_1|, |A_2| \le (1/2+\epsilon)n$, and $a_i \le \epsilon n$, $3 \le i \le k$,

(‡) $|E_{even}(A,B) \setminus E| \le \epsilon n^k$.

Note that we obtain (‡) after switching A_1 and B_1 if $|E_{odd}(A,B) \setminus E| \le \epsilon n^k$. Furthermore, the above two properties remain valid if we switch an even number of A_i with B_i. Thus we may switch A_1, A_i with B_1, B_i whenever $|A_i| > |B_i|$ for some $i \ge 3$. This results in $|A_i| \le |B_i|$ for all $i \ge 3$ eventually. Moreover, by Fact 2.4 (i) and (†), we know that $\delta_1(H) \ge (1/2-\epsilon)n^{k-1}$.

We now define *atypical* vertices. Let W be the set of $u \in V$ such that there exists an even $\mathbf{v} \in EV_k$ such that $u \in V^{\mathbf{v}}$ and $\overline{\deg}_{H(\mathbf{v})}(u) > \sqrt{\epsilon} n^{k-1}/2$. Let $W_0 := W \cap (V_1 \cup V_2)$. Since each vertex in W contribute at least $\sqrt{\epsilon} n^{k-1}/2$ k-sets towards $|E_{even}(A,B) \setminus E|$ (and such a k-set can be counted at most k times), by (‡), we have

$$|W_0| \le |W| \le \frac{k\epsilon n^k}{\sqrt{\epsilon} n^{k-1}/2} \le 2k\sqrt{\epsilon} n. \tag{5.2}$$

When forming a matching of H, we prefer to use the edges that intersect both A_1, A_2 or both B_1, B_2 – we will call them *horizontal edges*. Correspondingly, the edges that intersect both A_1, B_2 or both A_2, B_1 are *diagonal*. We distinguish the vertices of W_0 that lie in fewer horizontal edges from the rest of W_0. For $i = 1, 2$, let W_{A_i} be the set of vertices of $W_0 \cap A_i$ that lie in less than $\epsilon_0 n^{k-1}$ horizontal edges; similarly let W_{B_i} be the set of vertices of $W_0 \cap B_i$ that lie in less than $\epsilon_0 n^{k-1}$ horizontal edges.

Define $B_i^0 := (B_i \setminus W_{B_i}) \cup W_{A_i}$ for $i = 1, 2$ and $B_i^0 := B_i$ for $i \ge 3$. Let $A_i^0 := V_i \setminus B_i^0$ for all i. Let $A^0 := \bigcup_{i \in [k]} A_i^0$ and $B^0 := V \setminus A^0$. Finally, let

$$q := |B_2^0| - |B_1^0| = |B_2| - |B_1| + |W_{A_2}| + |W_{B_1}| - |W_{A_1}| - |W_{B_2}|.$$

By (†) and (5.2), $|q| \le 2\epsilon n + 2k\sqrt{\epsilon} n \le 3k\sqrt{\epsilon} n$. By relabelling V_1 and V_2 if necessary, we may assume that $q \ge 0$. Note that we still have $|A_i| \le |B_i|$ for all $i \ge 3$.

Our goal is to remove a small matching and possibly some crossing k-sets (non-edges) from H such that we can apply Theorem 5.1 to the remaining subgraph of H. To achieve the goal, we conduct the following five steps: we remove disjoint matchings $M_1, \dots, M_4$ in the first four steps and a balanced vertex set S_5 in the fifth step. For $1 \le j \le 4$, we define $A^j := A^{j-1} \setminus V(M_j)$, $B^j := B^{j-1} \setminus V(M_j)$, and $V^j := A^j \cup B^j$. Let $A^5 := A^4 \setminus S_5$, $B^5 := B^4 \setminus S_5$ and $V^5 := A^5 \cup B^5$. For $1 \le j \le 5$ and $1 \le i \le k$, define $A_i^j := A^j \cap V_i$, $B_i^j := B^j \cap V_i$, and $V_i^j := A_i^j \cup B_i^j$.

Step 1. Reducing the gap between $|B_1^0|$ *and* $|B_2^0|$. Our first matching M_1 is crucial for balancing the sizes of B_1^0 and B_2^0, and this is the only place that we need the exact codegree condition. Let $H_1 := H[A_1^0 \cup B_2^0 \cup V_3 \cup \cdots \cup V_k]$ and note that

$$\delta_{[k]\setminus\{1\}}(H_1) \ge a_1 - |B_1^0|, \quad \delta_{[k]\setminus\{2\}}(H_1) \ge a_2 - (n - |B_2^0|)$$

and $\delta_{[k]\setminus\{i\}}(H_1) \ge a_i$ for $3 \le i \le k$. So we have

$$\sum_{i=1}^k \delta_{[k]\setminus\{i\}}(H_1) \ge \sum_{i=1}^k a_i + |B_2^0| - |B_1^0| - n = q - n + \sum_{i=1}^k a_i.$$

By (†), we have $\sum_{i=1}^k a_i \le n + k\epsilon n$. Thus,

$$q - n + \sum_{i=1}^k a_i \le q + k\epsilon n \le 4k\sqrt{\epsilon} n < \min_{i \in [k]} |V(H_1) \cap V_i| - k.$$

If $q - n + \sum_{i=1}^k a_i > 0$, then Fact 1.3 provides a matching M' of size $q - n + \sum_{i=1}^k a_i$ in H_1. Let $M_1 := M'$ if $\sum_{i=1}^k a_i \le n$ and let $M_1 \subseteq M'$ be a (sub)matching of size q if $\sum_{i=1}^k a_i > n$. Otherwise let $M_1 := \emptyset$. So we have $|M_1| \le q \le 3k\sqrt{\epsilon} n$ in all cases.

Step 2. Cleaning V_1 *and* V_2. In this step we find a matching M_2 that covers all the remaining vertices of W_0 and uses the same amount of the vertices from A_1^0 and A_2^0. Let $W_0' := (W_{A_1} \cup W_{A_2} \cup W_{B_1} \cup W_{B_2}) \setminus V(M_1)$ and $W_0'' := W_0 \setminus (W_0' \cup V(M_1))$. We cover the vertices of W_0'' and W_0' as follows.

(1) By definition, each vertex $u \in W_0''$ lies in at least $\epsilon_0 n^{k-1}$ horizontal edges, i.e., those that intersect both A_1 and A_2, or intersect both B_1 and B_2. By (5.2), among these edges, at least $\epsilon_0 n^{k-1}/2$ horizontal edges do not intersect W, so they intersect both A_1^0 and A_2^0, or intersect both B_1^0 and B_2^0. We greedily cover the vertices of W_0'' by these disjoint horizontal edges.

(2) Since $\delta_1(H) \ge (1/2-\epsilon)n^{k-1}$, by definition, every vertex $u \in W_0'$ lies in at least $(1/2-\epsilon)n^{k-1} - \epsilon_0 n^{k-1}$ diagonal edges. By the definitions of $A_1^0, A_2^0, B_1^0, B_2^0$ and $\epsilon \ll \epsilon_0$, each $u \in W_0'$ lies in at least $(1/2-\epsilon)n^{k-1} - \epsilon_0 n^{k-1} - |W_0|n^{k-2} \ge \epsilon_0 n^{k-1}$ edges that intersect both A_1^0 and A_2^0, or both B_1^0 and B_2^0. We greedily cover the vertices of W_0' by such edges.

To see why the above process is possible, we note that when finding an edge for a vertex u, the number of vertices that we need to avoid is at most $|V(M_1)| + k|W_0| \le 3k^2\sqrt{\epsilon}n + 2k^2\sqrt{\epsilon}n = 5k^2\sqrt{\epsilon}n$ by (5.2) and $|M_1| \le 3k\sqrt{\epsilon}n$. Hence these vertices lie in at most $5k^2\sqrt{\epsilon}n^{k-1} < \epsilon_0 n^{k-1}/2$ crossing $(k-1)$-sets, so we can find an edge that covers u and avoids all the existing edges.

Let us bound $|B_2^2| - |B_1^2| = |B_2^0 \setminus V(M_1 \cup M_2)| - |B_1^0 \setminus V(M_1 \cup M_2)|$. By the definition of M_1, we have $|B_1^0 \cap V(M_1)| = 0$ and $|B_2^0 \cap V(M_1)| = |M_1|$. By the definition of M_2, we have $|B_1^0 \cap V(M_2)| = |B_2^0 \cap V(M_2)|$. Thus,

$$|B_2^2| - |B_1^2| = |B_2^0| - |B_1^0| - |M_1| = q - |M_1|.$$

Note that

$$q - |M_1| = \begin{cases} n - \sum_{i=1}^k a_i & \text{if } q - n + \sum_{i=1}^k a_i \ge 0 \text{ and } \sum_{i=1}^k a_i \le n; \\ 0 & \text{if } q - n + \sum_{i=1}^k a_i \ge 0 \text{ and } \sum_{i=1}^k a_i > n; \\ q \le n - \sum_{i=1}^k a_i & \text{if } q - n + \sum_{i=1}^k a_i < 0. \end{cases}$$

So we have

$$0 \le |B_2^2| - |B_1^2| = q - |M_1| \le \max\left\{0, n - \sum_{i=1}^k a_i\right\} \le k-3. \tag{5.3}$$

Step 3. Cleaning $V_3, \dots, V_k$. Let X consist of all $x \in A_i \setminus W$ for some $i \ge 3$ such that $|A_i| \le 2\epsilon_0 n$. In this step we build a matching M_3 which covers all the remaining vertices of W and the vertices of X and satisfies that

$$-1 \le |B_1^2 \cap V(M_3)| - |B_2^2 \cap V(M_3)| \le 0. \tag{5.4}$$

Let W' be the set of vertices in $(W_3 \cup \dots \cup W_k) \setminus V(M_1 \cup M_2)$ that are contained in at least $3k^2\epsilon_0 n^{k-1}$ horizontal edges. Let $W'' := (W_3 \cup \dots \cup W_k) \setminus (V(M_1 \cup M_2) \cup W')$. Since $\delta_1(H) \ge (1/2-\epsilon)n^{k-1}$, by definition, each $u \in W''$ is contained in at least $(1/2-\epsilon)n^{k-1} - 3k^2\epsilon_0 n^{k-1}$ diagonal edges. Note that by (†), we have $|B_1||A_2| \le (1/4+3\epsilon)n^2$. Then since u lies in at most $|B_1||A_2|n^{k-3} \le (1/4+3\epsilon)n^{k-1}$ edges that intersect both B_1 and A_2, there are at least $3k^2\epsilon_0 n^{k-1}$ edges that contain u and intersect both A_1 and B_2. Note that by symmetry, the same statement holds for u, A_2 and B_1. Finally, for any vertex $x \in X$, assume that $x \in A_i$ for some $3 \le i \le k$. Since the binary vectors $\mathbf{v} \in \{0,1\}^k$ with exactly two 1's are even, the fact that $x \notin W$ implies that x is contained in at least

$$\prod_{j \in [k] \setminus \{1,i\}} |B_j| \cdot |A_1| - \frac{1}{2}\sqrt{\epsilon}n^{k-1} \ge \frac{n^{k-1}}{2^k} - \frac{1}{2}\sqrt{\epsilon}n^{k-1} > 3k^2\epsilon_0 n^{k-1}$$

edges in $A_1 \cup (\bigcup_{2 \le j \le k, j \ne i} B_j) \cup A_i$, and in at least $3k^2\epsilon_0 n^{k-1}$ edges in $B_1 \cup A_2 \cup (\bigcup_{3 \le j \le k, j \ne i} B_j) \cup A_i$, where we used $|A_1||B_2| \ge n^2/8$ and $|B_i| \ge n/2$ for $3 \le i \le k$ in the first inequality.

(1) We first greedily find $|W'|$ disjoint horizontal edges such that each of them contains one vertex of W' and no other vertices from $W \cup X \cup V(M_1 \cup M_2)$.

(2) Next, we split $W'' \cup X$ arbitrarily to W_1'' and W_2'' of sizes $\lfloor |W'' \cup X|/2 \rfloor$ and $\lceil |W'' \cup X|/2 \rceil$, respectively. We greedily find $|W_1''|$ disjoint edges such that each of them contains one vertex $u \in W_1''$, one vertex from each of B_1 and A_2, and no other vertices from $W \cup X \cup V(M_1 \cup M_2)$; moreover, if $u \in A_i \subseteq X$, then the edge is taken in $B_1 \cup A_2 \cup (\bigcup_{3 \le j \le k, j \ne i} B_j) \cup A_i$.

Finally, we greedily find $|W_2''|$ disjoint edges such that each of them contains one vertex $u \in W_2''$, one vertex from each of A_1 and B_2, but no other vertices from $W \cup X \cup V(M_1 \cup M_2)$; moreover, if $u \in A_i \subseteq X$, then the edge is taken in $A_1 \cup (\bigcup_{2 \le j \le k, j \ne i} B_j) \cup A_i$.

The above process is possible because when considering a vertex u, the number of vertices that we need to avoid is at most $|V(M_1)|+|V(M_2)|+k|W|+k\cdot 2(k-2)\epsilon_0 n \le |V(M_1)|+2k|W|+k\cdot 2(k-2)\epsilon_0 n < 3k^2\epsilon_0 n$ because of (5.2), the facts $|M_1|\le 3k\sqrt{\epsilon}n$ and $|X|\le 2(k-2)\epsilon_0 n$. Hence these vertices lie in less than $3k^2\epsilon_0 n^{k-1}$ $(k-1)$-sets, so we can always find a desired edge that covers u and avoids all the existing edges. Let M_3 be the matching obtained in this step. Note that (5.4) holds by construction.

Step 4. Balancing the sizes of B_1^3 and A_2^3. Let $m := |B_1^3|-|A_2^3|$. We find a matching M_4 of size $|m|$ as follows. If $m\ge 0$, then M_4 consists of m disjoint edges from B^3 that are disjoint from $M_1\cup M_2\cup M_3$. Since $(0,\dots,0)\in EV_k$, this can be done since $H[(0,\dots,0)]$ is almost complete. Otherwise M_4 consists of $|m|$ disjoint edges with location vector $(1,1,0,\dots,0)$ that are disjoint from $M_1\cup M_2\cup M_3$ – this is possible because $H[(1,1,0,\dots,0)]$ is almost complete.

After removing M_4, the resulting sets B_1^4 and A_2^4 satisfy $|B_1^4|=|A_2^4|$. The definition of M_4 implies that $|B_1^2\cap V(M_4)|=|B_2^2\cap V(M_4)|$. Together with (5.3) and (5.4), this gives

$$-1\le |B_2^4|-|B_1^4| = |B_2^2|-|B_1^2|+|B_1^2\cap V(M_3)|-|B_2^2\cap V(M_3)|\le k-3.$$

Step 5. Balancing the sizes of B_1^4 and B_2^4. Let $t:=|B_2^4|-|B_1^4|$ and thus $-1\le t\le k-3$. If $t>0$, then $n-\sum_{i=1}^k a_i\ge t>0$. Let S_5 be a kt-set in V^4 with t vertices from each of A_1^4, B_2^4, and V_i^4 for $3\le i\le k$ such that $|A^4\setminus S_5|$ is even. The requirement that $|A^4\setminus S_5|$ is even can be easily fulfilled if any A_i^4, $i\ge 3$, is not empty. On the other hand, if all $A_3^4,\dots,A_k^4$ are empty, then since $|B_2^4\setminus S_5|=|B_1^4|$, we have $|A_1^4\setminus S_5|=|A_2^4|$ and consequently,

$$|A^4\setminus S_5| = |A_1^4\setminus S_5|+|A_2^4| = 2|A_2^4|$$

is even. Since $t\le n-\sum_{i=1}^k a_i$, to complete the proof, it suffices to find a perfect matching in $V^5:=V^4\setminus S_5$. If $t=-1$, then let S_5 be a k-set with one vertex from each of B_1^4, A_2^4 and V_i^4 for $3\le i\le k$ such that $|A^4\setminus S_5|$ is even – this can be achieved by the same argument as in the $t>0$ case. Again it suffices to find a perfect matching in V^5. At last, if $t=0$ then set $S_5=\emptyset$. In this case $|A^5|=|A^4\setminus S_5|$ may be odd; however, it suffices to find a matching of size $|V^5|-1$ in $H[V^5]$. In summary, it remains to find a perfect matching in $H[V^5]$ if $|A^5|$ is even and a matching of size $|V^5|-1$ otherwise. This will follow from Theorem 5.1 after we verify its assumptions.

Let $n':=|V_1^5|$ and $H':=H[V^5]$. Note that

$$|M_1\cup M_2\cup M_3|\le |M_1|+|W|+|X|\le 3k\sqrt{\epsilon}n+2k\sqrt{\epsilon}n+2k\epsilon_0 n, \quad \text{and}$$

$$|M_4| = ||B_1^3|-|A_2^3|| \le |B_1|-|A_2|+|M_1\cup M_2\cup M_3|\le 3k\epsilon_0 n,$$

where $|B_1|-|A_2|\le 2\epsilon n$ by (†). Note that we have $V(M_4)\cap A_i=\emptyset$ for $3\le i\le k$, and when building M_3, we may use the vertices of A_i, $3\le i\le k$, of size at least $2\epsilon_0 n$ only when we cover the vertices of W. Thus for $3\le i\le k$, if $|A_i^5|\ne 0$, then

$$\begin{aligned}|A_i^5| &\ge |A_i|-|M_1|-|V(M_2\cup M_3)\cap A_i|-|S_5\cap A_i|\\ &\ge 2\epsilon_0 n-3k\sqrt{\epsilon}n-2k\sqrt{\epsilon}n-(k-3)\ge \epsilon_0 n\ge \epsilon_0 n',\end{aligned}$$

and $|A_i^5|\le |A_i|\le n/2<(1-\epsilon_0)n'$. Moreover, by the choice of M_4 and S_5, we have $|A_2^5|=|B_1^5|=|B_2^5|$, and thus $|A_1^5|=|A_2^5|=n'/2$. In particular, this means that n' is even. Finally, note that

$$\begin{aligned}n' &= n-|M_1|-|M_2|-|M_3|-|M_4|-|S_5|/k\\ &\ge n-3k\sqrt{\epsilon}n-2k\sqrt{\epsilon}n-2k\epsilon_0 n-3k\epsilon_0 n-(k-3)\ge (1-6k\epsilon_0)n.\end{aligned}$$

So for any $\mathbf{v}\in EV_k$, and any vertex $u\in H'(\mathbf{v})$, since $u\notin W$ and n is large enough, we have

$$\overline{\deg}_{H'(\mathbf{v})}(u)\le \sqrt{\epsilon}n^{k-1}/2<\sqrt{\epsilon}n'^{k-1}.$$

So we are done by Theorem 5.1 with $\eta=\sqrt{\epsilon}$. □

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Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508-090, São Paulo, Brazil

E-mail address, Jie Han: jhan@ime.usp.br

Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303

E-mail address, Chuanyun Zang: chuanyun.zang@gmail.com

E-mail address, Yi Zhao: yzhao6@gsu.edu