Perfect Matchings in 4-uniform hypergraphs

Imdadullah Khan

Department of Computer Science
Rutgers University
New Brunswick, NJ, USA 08903
`imdadk@cs.rutgers.edu`

January 10, 2011

Abstract

A perfect matching in a 4-uniform hypergraph is a subset of $\lfloor \frac{n}{4} \rfloor$ disjoint edges. We prove that if H is a sufficiently large 4-uniform hypergraph on $n = 4k$ vertices such that every vertex belongs to more than $\binom{n-1}{3} - \binom{3n/4}{3}$ edges then H contains a perfect matching. This bound is tight and settles a conjecture of Hán, Person and Schacht.

1 Introduction and Notation

For graphs we follow the notation in [1]. For a set T, we refer to all of its k-element subsets (k-sets for short) as $\binom{T}{k}$ and the number of such k-sets as $\binom{|T|}{k}$. We say that $H = (V(H), E(H))$ is an r-uniform hypergraph or r-graph for short, where $V(H)$ is the set of vertices and $E \subset \binom{V(H)}{r}$ is a family of r-sets of $V(H)$. When the graph referred to is clear from the context we will use V instead of $V(H)$ and will identify H with $E(H)$. For an r-graph H and a set $D = \{v_1, \ldots, v_d\} \in \binom{V}{d}, 1 \le d \le r$, the degree of D in H, $deg_H(D) = deg_r(D)$ denotes the number of edges of H that contain D. For $1 \le d \le r$, let

$$\delta_d = \delta_d(H) = \min \left\{ deg_r(D) \ : \ D \in \binom{V}{d} \right\}$$

We say that $H(V_1, \ldots, V_r)$ is an r-partite r-graph, if there is a partition of $V(H)$ into r sets, i.e. $V(H) = V_1 \cup \cdots \cup V_r$ and every edge of H uses exactly one vertex from each V_i. We call it a balanced r-partite r-graph if all V_i's are of the same size. We say $H(V_1, \ldots, V_r)$ is a complete r-partite r-graph if every r-tuple that uses one vertex from each V_i belongs to $E(H)$. We denote a complete balanced r-partite r-graph by $K^{(r)}(t)$, where $t = |V_i|$. For $r = 3$, we refer to the balanced 3-partite 3-graph $H(V_1, V_2, V_3)$, where $|V_i| = 4$ as a $4 \times 4 \times 4$ 3-graph.

For an r-graph H, when A and B make a partition of $V(H)$, for a vertex $v \in A$ we denote by $deg_r\left(v, \binom{B}{r-1}\right)$ the number of $(r-1)$-sets of B that make edges with v while $e_r\left(A, \binom{B}{r-1}\right)$ is the sum of $deg_r\left(v, \binom{B}{r-1}\right)$ over all $v \in A$ and $d_r\left(A, \binom{B}{r-1}\right) = e_r\left(A, \binom{B}{r-1}\right) / |A| \binom{|B|}{r-1}$. We denote by $H\left(A, \binom{B}{r-1}\right)$, such an r-graph, when all edges use one vertex from A and $r-1$ vertices from B. Similarly $H\left(A, B, \binom{C}{r-2}\right)$ is an r-graph where A, B and C make a partition of $V(H)$, and every edge in H uses one vertex each from A and B and $r-2$ vertices from C. Degrees of vertices in A and B are similarly defined as above. The density of $H\left(A, B, \binom{C}{r-2}\right)$ is

$$d_r\left(A, B, \binom{C}{r-2}\right) = \frac{\left| E\left(H\left(A, B, \binom{C}{r-2}\right)\right)\right|}{|A||B|\binom{|C|}{r-2}}$$

When $A_1, \ldots, A_r$ make a partition of $V(H)$, for a vertex $v \in A_1$ we denote by $deg_r(v, (A_2 \times \cdots \times A_r))$ the number of edges in the r-partite r-graph induced by subsets $\{v\}, A_2, \ldots, A_r$, and $e(A_1, (A_2 \times \cdots \times A_r))$ is the sum of $deg_r(v, (A_2 \times \cdots \times A_r))$ over all $v \in A_1$. Similarly

$$d_r(A_1, (A_2 \times \cdots \times A_r)) = \frac{e(A_1, (A_2 \times \cdots \times A_r))}{|A_1 \times A_2 \times \cdots \times A_r|}$$

An r-graph H on n vertices is η-*dense* if it has at least $\eta\binom{n}{r}$ edges. We use the notation $d_r(H) \geq \eta$ to refer to an η-*dense* r-graph H. For $U \subset V$, $H|_U$ is the restriction of H to U. For simplicity we refer to $d_r(H|_U)$ as $d_r(U)$ and to $E(H|_U)$ as $E(U)$. A matching in H is a set of disjoint edges of H and a perfect matching is a matching that contains all vertices. Moreover we will only deal with r-graphs on n vertices where $n = rk$ for some integer k, we denote this by $n \in r\mathbb{Z}$. Throughout the paper log denotes the base 2 logarithm.

Definition 1. *Let d, r and n be integers such that $1 \leq d < r$, and $n \in r\mathbb{Z}$. Denote by $m_d(r,n)$ the smallest integer m, such that every r-graph H on n vertices with $\delta_d(H) \geq m$ contains a perfect matching.*

For graphs $(r = 2)$, by the Dirac theorem on Hamiltonicity of graphs, it s easy to see that $m_1(2, n) \leq n/2$, and since the complete bipartite $K_{n/2-1, n/2+1}$ does not have a perfect matching we get $m_1(2, n) = n/2$. For $r \geq 3$ and $d = r - 1$, it follows from a result of Rödl, Ruciński and Szemerédi on Hamiltonicity of r-graph [10] that $m_{r-1}(r, n) \leq n/2 + o(n)$. Kühn and Osthus [5] improved this result to $m_{r-1}(r, n) \leq n/2 + 3r^2\sqrt{n \log n}$. This bound was further sharpened in [9] to $m_{r-1}(r, n) \leq n/2 + C \log n$. In [11] the bound was improved to almost the true value, it was proved that $m_{r-1}(r, n) \leq n/2 + r/4$. Finally [12] settled the problem for $d = r - 1$.

The case $d < r - 1$ is rather hard, in [7] it is proved that for all $d \geq r/2$, $m_d(r, n)$ is close to $\frac{1}{2}\binom{n-d}{r-d}$. For $1 \leq d < r/2$ in [2] it was proved that

$$m_d(r, n) \leq \left(\frac{r - d}{r} + o(1)\right)\binom{n - d}{r - d}$$

A recent survey of these and other related results appear in [15]. In [2] the authors posed the following conjecture.

Conjecture 1 ([2]). *For all $1 \leq d < r/2$,*

$$m_d(r, n) \sim \max\left\{\frac{1}{2}, 1 - \left(\frac{r - 1}{r}\right)^{r-d}\right\}\binom{n - d}{r - d}$$

Note that for $r = 4$ and $d = 1$ the above bound yields

$$m_1(3, n) \sim \frac{37}{64}\binom{n - 1}{3}$$

For $r = 3$ and $d = 1$ this conjecture was proved in [13] and [6]. Markstöm and Ruciński in [14] improved the bound on $m_d(r, n)$ for $1 \leq d < r/2$ slightly, by proving

$$m_d(r, n) \leq \left(\frac{r - d}{r} - \frac{1}{r^{r-d}} + o(1)\right)\binom{n - d}{r - d}$$

Furthermore for $r = 4$ and $d = 1$ in [14] the authors proved that $m_1(4, n) \leq \left(\frac{42}{64} + o(1)\right)\binom{n-1}{3}$. In this paper we settle Conjecture 1 for the case $r = 4$ and $d = 1$. The main result in this paper is the following theorem.

Theorem 1. *There exist an integer n_0 such that if H is a 4-graph on $n \geq n_0$ ($n \in 4\mathbb{Z}$) vertices, and*

$$\delta_1(H) \geq \binom{n - 1}{3} - \binom{3n/4}{3} + 1 \tag{1}$$

then H has a perfect matching.

On the other hand the following construction from [2] shows that this result is tight.

Construction 1. *Let A and B be disjoint sets with $|A| = \frac{n}{4} - 1$ and $|B| = n - |A|$. Let $H = (V(H), E(H))$ be a 4-graph such that $V(H) = A \cup B$ and $E(H)$ is the set of all 4-tuples of vertices, T, such that $|T \cap A| \geq 1$.*

We have $\delta_1(H) = \binom{n-1}{3} - \binom{3n/4}{3}$ (the degree of a vertex in B) but since every edge in a matching must use at least one vertex from A, the maximum matching in this graph is of size $|A| = \frac{n}{4} - 1$
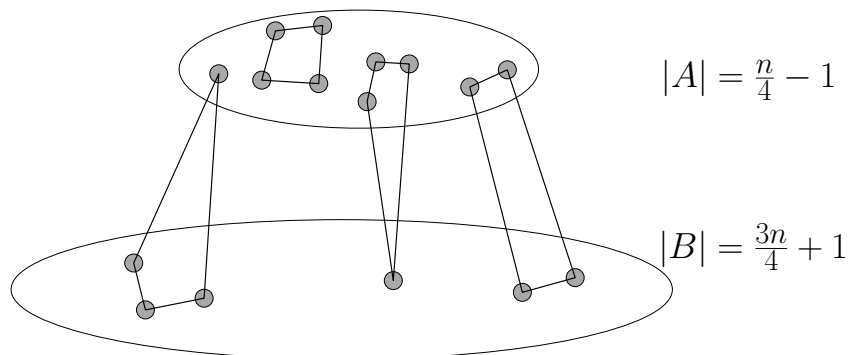
Figure 1: The extremal example: A Quadrilateral represent an edge. Every edge intersects the set A.

1.1 Outline of the proof

We distinguish two cases to prove Theorem 1. When H is *'far from'* the extremal example as in Construction 1 we use the *absorbing lemma* to find a perfect matching in H. The absorbing lemma (Lemma 10) roughly states, that any 3-graph H satisfying (1), contains a small matching M with the property that every "not too large" subset of vertices can be absorbed into M. In Section 4, we first remove an absorbing matching M from H. Then using the tools developed in section 2 and the auxiliary results of section 3, we find an *almost perfect matching* in $H|_{V\setminus V(M)}$. The left over vertices are absorbed into M to get a perfect matching in H. We will show that we can either find an almost perfect matching or H is in the extremal case.

For a constant $0 < \alpha < 1$, we say that H is *α-extremal*, if the following is satisfied, otherwise it is *α-non-extremal*.

Definition 2 (Extremal Case with parameter α). *There exists a $B \subset V(H)$ such that*

- $|B| \geq \left(\frac{3}{4} - \alpha\right) n$

- $d_4(B) < \alpha$.

In section 5, using a König-Hall type argument, we build a perfect matching in H, when it is α-extremal.

2 Main tools

We use the following two easy observations that will, nevertheless, be useful later on.

4

Fact 2. *If $G(A, B)$ is a 2η-dense bipartite graph, then there must be at least $\eta|B|$ vertices in B for which the degree in A is at least $\eta|A|$.*

Indeed, otherwise the total number of edges would be less than

$$\eta|A||B| + \eta|A||B| = 2\eta|A||B|$$

a contradiction to the fact that $G(A, B)$ is 2η-dense.

Fact 3. *If $G(A, B)$ is a bipartite graph, $|A| = c_1 \log n$, $|B| = c_2 n$ and for every vertex $b \in B$ $deg(b, A) \geq \eta|A|$, then we can find a complete bipartite subgraph $G'(A', B')$ of G such that $A' \subset A, B' \subset B, |A'| \geq \eta|A|$ and $|B'| \geq c_2 n^{(1-c_1)}$.*

To see this consider the neighborhoods in A, of the vertices in B. Since there can be at most $2^{|A|} = n^{c_1}$ such neighborhoods, by averaging there must be a neighborhood that appears for at least $\frac{c_2 n}{n^{c_1}} = c_2 n^{(1-c_1)}$ vertices of B. This means that we can find the desired complete bipartite graph.

The main tool in this paper is the following result of Erdös [4], on complete balanced r-partite subhypergraphs of r-graphs.

Lemma 4. *For every integer $l \geq 1$ there is an integer $n_0 = n_0(r, l)$ such that: Every r-graph on $n > n_0$ vertices, that has at least $n^{r-1/l^{r-1}}$ edges, contains a $K^{(r)}(l)$.*

In particular Lemma 4 implies that for $\eta > 0$ and a sufficiently small constant β, if H is an r-graph on $n > n_0(\eta, \beta)$ vertices with

$$|E(H)| \geq \eta \binom{n}{r}$$

then H contains a $K^{(r)}(t)$, where

$$t = \beta(\log n)^{1/r-1}.$$

This is so because $\eta\binom{n}{r} \geq \frac{n^r}{n^{1/\beta^{r-1} \log n}} = \frac{n^r}{2^{1/\beta^{r-1}}}$, when $\eta > \frac{2r!}{2^{1/\beta^{r-1}}}$.

The following three lemmas are repeatedly used in section 4.

Lemma 5. *Let $H(A, \binom{B}{3})$ be a 4-graph, $|A| = c_1 m$, $|B| = c_2 2^{m^3}$, for some constants $0 < c_1, c_2 < 1$, if*

$$d_4\left(A, \binom{B}{3}\right) \geq 2\eta$$

then there exists a complete 4-partite 4-graph $H'(A_1, B_1, B_2, B_3)$, with $A_1 \subset A$ and B_1, B_2 and B_3 are disjoint subsets of B such that $|A_1| = |B_1| = |B_2| = |B_3| = \eta|A|$.

Proof. First apply Fact 2 to get a subset of 3-sets of vertices in B, $T_1 \subset \binom{B}{3}$, such that every 3-set in T_1 makes edges with at least $\eta|A|$ vertices in A and $|T_1| \geq \eta\binom{|B|}{3} \geq \eta|B|^3/10$. Next we find a $T_2 \subset T_1$, such that all 3-sets in T_2 make edges with the same subset of A (say $A_1 \subset A$). By Fact 3 we have

$$|T_2| \geq \frac{\eta|B|^3/10}{2^{c_1 m}} = \frac{\eta}{10}\frac{\left(c_2 2^{m^3}\right)^3}{2^{c_1 m}} = \frac{c_2^3\eta}{10}2^{m^3(3-c_1/m^2)} = \frac{c_2^3\eta}{10}\left(\frac{|B|}{c_2}\right)^{3-c_1/m^2} \geq |B|^{3-1/(\eta|A|)^2}$$

where the last inequality follows when m is sufficiently large and η, c_1 and c_2 are small constants.

Now construct an auxiliary 3-graph H_1 where $V(H_1) = B$ and edges of H_1 corresponds to 3-sets in T_2. Applying lemma 4 on H_1 (for $r = 3$) we get a complete 3-partite 3-graph with color classes B_1, B_2 and B_3 each of size $\eta|A|$. Clearly A_1, B_1, B_2 and B_3 corresponds to color classes of a complete 4-partite 4-graph in H as in the statement of the lemma. $\square$

Lemma 6. *Let $H(A, B, \binom{Z}{2})$ be a 4-graph with $|A| = |B| = c_1 m$, $|Z| = c_2 2^{m^3}$ for some constants $0 < c_1, c_2 < 1$, if*

$$d_4\left(A, B, \binom{Z}{2}\right) \geq 2\eta$$

then there exists a complete 4-partite 4-graph $H'(A', B', Z_1, Z_2)$, with $A' \subset A$, $B' \subset B$ and Z_1 and Z_2 are disjoint subsets of Z such that $|A'| = |B'| = |Z_1| = |Z_2| = c\log m$ for a constant $c = c(c_1, c_2, \eta)$.

Proof. First apply Fact 2 to get a subset of pairs of vertices in Z, $P_1 \subset \binom{Z}{2}$, such that every pair in P_1 makes edges with at least $\eta|A||B|$ pairs of vertices in $A \times B$ and $|P_1| \geq \eta\binom{|Z|}{2} \geq \eta|Z|^2/3$. Next we find a $P_2 \subset P_1$, such that all pairs in P_2 make edges with the same subset of pairs in $A \times B$ (say $Q_1 \subset A \times B$). By Fact 3 we have

$$|P_2| \geq \frac{\eta|Z|^2/3}{2^{(c_1 m)^2}} = \frac{\eta}{3}\frac{\left(c_2 2^{m^3}\right)^2}{2^{(c_1 m)^2}} = \frac{c_2^2\eta}{3}2^{m^3(2-c_1^2/m)} = \frac{c_2^2\eta}{3}\left(\frac{|Z|}{c_2}\right)^{2-c_1^2/m} \geq |Z|^{2-1/(\log m)}$$

where the last inequality follows when m is sufficiently large, and η, c_1 and c_2 are small constants.

Now construct an auxiliary 2-graph G_1 where $V(G_1) = Z$ and edges of G_1 corresponds to pairs in P_2. Applying lemma 4 on G_1 (for $r = 2$) we get a complete bipartite graph with color classes Z_1 and Z_2 each of size $\log m$.

Similarly construct an auxiliary bipartite graph G_2 with color classes A and B, and edges of G_2 corresponds to pairs in Q_1. Since we have $|Q_1| \geq \eta|A||B|$ applying lemma 4 on G_2 (for $r = 2$) we

get a complete bipartite graph with color classes A' and B', each of size $c \log m$. Clearly A', B' and a subset of vertices each from Z_1 and Z_2 of size $|A'|$ corresponds to color classes of the balanced complete 4-partite 4-graph in H as in the statement of the lemma. □

Lemma 7. *Let $H(A, B, C, Z)$ be a 4-partite 4-graph with $|A| = |B| = |C| = c_1 m$ and $|Z| = c_2 2^{m^3}$ for some constants $0 < c_1, c_2 < 1$. If*

$$d_4(Z, (A \times B \times C)) \geq 2\eta$$

then there exists a complete 4-partite 4-graph $H'(A', B', C', Z')$ such that $|A'| = |B'| = |C'| = |Z'| = \beta\sqrt{\log |A|}$.

Proof. First apply Fact 2 on H to get a subset Z_1 of Z such that for every vertex $z \in Z_1$, $deg_3(z, (A \times B \times C)) \geq \eta|A|^3$ and $|Z_1| \geq \eta|Z|$. Now consider the auxiliary bipartite graph $G(D, Z_1)$, where $D = A \times B \times C$ and a vertex $z \in Z_1$ is connected to a 3-set $(a, b, c) \in D$ if (a, b, c, z) makes an edge of H. An application of Fact 3 on G gives a complete bipartite graph $G_2(D', Z_1')$ where

$$|D'| \geq \eta|A|^3 \quad \text{and} \quad Z_1' \geq \eta c_2 2^{m^3(1-c_1^3)} > |A|$$

when m is sufficiently large.

Let G_3 be a 3-partite 3-graph where the color classes are A, B and C and edges correspond to 3-sets in D'. Since $|D'| \geq \eta|A|^3$ applying Lemma 4 on G_3 ($r = 3$), we get a balanced complete 3-partite 3-graph in G_3 with color classes A', B' and C', such that $|A'| = |B'| = |C'| \geq \beta\sqrt{\log |A|}$. Clearly A', B', C' and a subset of Z_1' of size $|A'|$, correspond to the color classes of the required complete 4-partite 4-graph. □

We will frequently apply the following folklore statements (proofs are omitted):

Lemma 8. *Every graph H has a subgraph H' such that $\delta_1(H') \geq |E(H)|/|V(H)|$.*

Lemma 9. *Every 3-graph H on n vertices, with $\delta_1(H) \geq \eta\binom{n}{3}/n$, has a matching of size $\eta n/24$.*

Finally we use the following lemma from [2].

Lemma 10. *(Absorbing Lemma) For every $\eta > 0$, there is an integer $n_0 = n_0(\eta)$ such that if H is a 4-graph on $n \geq n_0$ vertices with $\delta_1(H) \geq (1/2 + 2\eta)\binom{n}{3}$, then there exist a matching M in H of size $|M| \leq \eta^4 n/4$ such that for every set $W \subset V \setminus V(M)$ of size at size at most $\eta^8 n \geq |W| \in 4\mathbb{Z}$, there exists a matching covering exactly the vertices in $V(M) \cup W$.*

3 Auxiliary results

For a $4 \times 4 \times 4$ 3-graph, denote by Q_1, Q_2, Q_3 its 3 color classes and let $Q_1 = \{a_1, a_2, a_3, a_4\}$, $Q_2 = \{b_1, b_2, b_3, b_4\}$ and $Q_3 = \{c_1, c_2, c_3, c_4\}$. For two vertices, $x \in Q_i$, $y \in Q_j$, $i \neq j$, $N(x, y)$ denotes the neighborhood of the pair x, y, i.e. the set of vertices in the third color class that make edges with the pair x, y. Let $deg(x, y) = |N(x, y)|$. For two disjoint pairs (x_1, y_1) and (x_2, y_2) in $Q_i \times Q_j$, $i \neq j$ the pairs (x_1, y_2) and (x_2, y_1) are called the crossing pairs and the value $deg(x_1, y_2) + deg(x_2, y_1)$ is referred to as the *crossing degree sum*. We define the following four special $4 \times 4 \times 4$ 3-graphs.

Definition 3. H_{432} *is a* $4 \times 4 \times 4$ *3-graph, such that there exist 3 disjoint pairs in a* $Q_i \times Q_j$, $i \neq j$, *with degrees at least 4, 3 and 2 respectively.*

In the figures the triplets joined by a line represent an edge in the 3-graph. Let the H_{432} be as in Figure 2(a) or Figure 2(b). If the H_{432} does not have a perfect matching then we must have $deg(b_4, c_4) = 0$, (as otherwise by the König-Hall criteria we get a perfect matching).
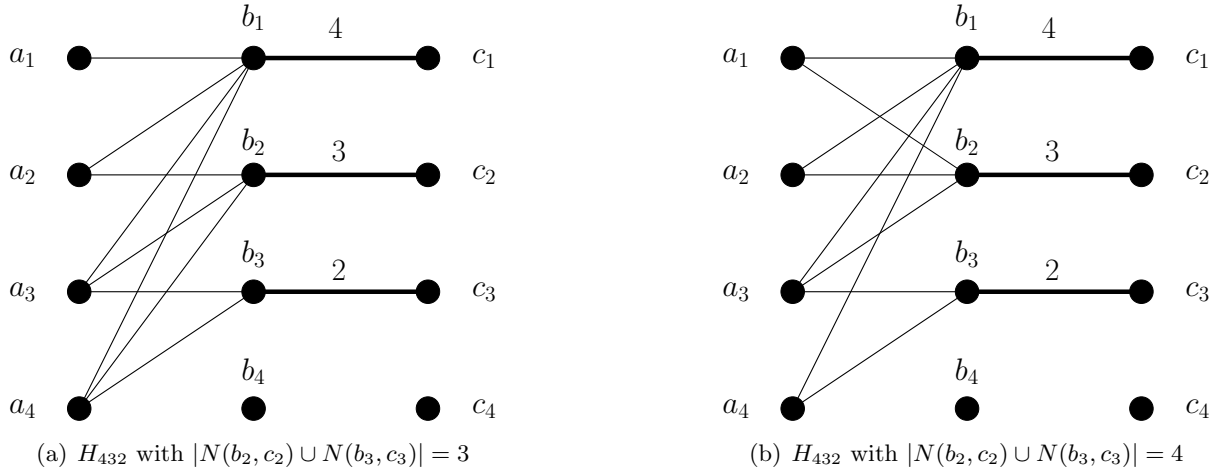

(a) H_{432} with $|N(b_2, c_2) \cup N(b_3, c_3)| = 3$ (b) H_{432} with $|N(b_2, c_2) \cup N(b_3, c_3)| = 4$

Figure 2: The two types of H_{432}. Labels on edges is the degree of the pair.

Definition 4. H_{4221} *is a* $4 \times 4 \times 4$ *tripartite 3-graph, such that there exist 4 disjoint pairs in a* $Q_i \times Q_j$, $i \neq j$, *with degrees at least 4, 2, 2 and 1 respectively.*

Let the H_{4221} be as in Figure 3(a). If $|N(b_2, c_2) \cup N(b_3, c_3) \cup N(b_4, c_4)| = 3$ then again by the König-Hall criteria we get that H_{4221} has a perfect matching. We only consider the H_{4221} that has no perfect matching.

Definition 5. H_{3321} is a $4 \times 4 \times 4$ 3-graph, such that there exist 4 disjoint pairs in a $Q_i \times Q_j$, $i \neq j$, with degrees at least 3, 3, 2 and 1 respectively (see Figure 3(b)).

(a) H_{4221} with no perfect matching

(b) H_{3321} with no perfect matching
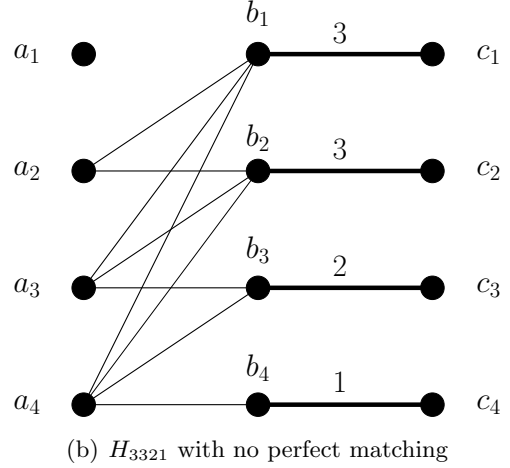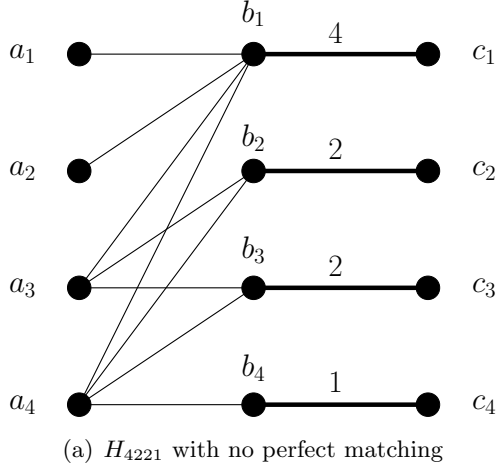
Definition 6. H_{ext} is a $4 \times 4 \times 4$ 3-graph with exactly 37 edges such that there are three vertices, one in each of Q_1, Q_2 and Q_3, and all edges are incident to at least one of these three vertices.

The following lemma that classifies, $4 \times 4 \times 4$ 3-graphs with at least 37 edges, will be very useful in the subsequent section.

Lemma 11. Let $H(Q_1, Q_2, Q_3)$ be a $4 \times 4 \times 4$ 3-graph. If $|E(H)| \geq 37$ then one of the following must be true

1. H has a perfect matching.

2. H has a subgraph isomorphic to H_{3321}

3. H has a subgraph isomorphic to H_{432}

4. H has a subgraph isomorphic to H_{4221}

5. H is isomorphic to H_{ext}.

Proof. We consider the following cases based on degree of pairs in $Q_i \times Q_j$.

Case 1: There is Q_i and Q_j, $i \neq j$ such that no pair in $Q_i \times Q_j$ has degree 4.

9

Let Q_2, Q_3 be such a pair. Since $|E(H)| \geq 37$ and no pair has degree 4, at least 5 out of the 16 pairs in $Q_2 \times Q_3$ must be of degree 3. Which implies that there must be at least 2 disjoint pairs of degree 3. Consider the largest set of disjoint pairs of degree 3 in $Q_2 \times Q_3$. Assume that there are 3 disjoint pairs in $Q_2 \times Q_3$ of degree 3, say (b_1, c_1), (b_2, c_2) and (b_3, c_3).

If $deg(b_4, c_4) \geq 1$, then we have a H_{3321}. So assume that $deg(b_4, c_4) = 0$, The total number of edges incident to pairs in $\{b_1, b_2, b_3\} \times \{c_1, c_2, c_3\}$ is at most 27 (as there are 9 pairs and degree of every pair is at most 3), the remaining at least 10 edges are incident to either b_4 or c_4. Which implies that there must be at least one pair (say (b_3, c_3)), such that crossing degree sum of (b_3, c_3) and (b_4, c_4) is at least 4. Therefore we have that the degree of one crossing pair is at least 2 and that of the other crossing pair is at least 1. These two crossing pairs together with (b_1, c_1) and (b_2, c_2) gives us a subgraph isomorphic to H_{3321}.

On the other hand, if there are exactly 2 disjoint pairs of degree 3, say (b_1, c_1) and (b_2, c_2). Again the total number of edges incident to pairs in $\{b_1, b_2\} \times \{c_1, c_2\}$ is at most 12 (as there are 4 pairs and degree of every pair is at most 3). If there is a pair in $\{b_3, b_4\} \times \{c_3, c_4\}$ (say (b_3, c_3)) such that the *crossing degree sum* of (b_1, c_1) and (b_3, c_3) is at least 6, then since there is no degree 4 pair we must have that both $deg(b_1, c_3)$ and $deg(c_1, b_3)$ are 3. Now these crossing pairs together with (a_2, b_2) are 3 disjoint pairs of degree 3 which is a contradiction to the maximality of the set of disjoint pairs of degree 3. Therefore we must have that the sum of degrees of pairs in $\{b_1, b_2\} \times \{c_3, c_4\}$ and $\{c_1, c_2\} \times \{b_3, b_4\}$ is at most $4 \times 5 = 20$. Hence the number of edges of H incident to pairs in $\{b_3, b_4\} \times \{c_3, c_4\}$ is at least $37 - 12 - 20 = 5$ and no pair has degree 3. Therefore, in $\{b_3, b_4\} \times \{c_3, c_4\}$, we can find two disjoint disjoint pairs, (say (b_3, c_3) and (b_4, c_4)) with degree at least 2 and 1 respectively and we get a graph isomorphic to H_{3321}.

Case 2: There is a Q_i and Q_j, $i \neq j$ such that exactly one disjoint pair in $Q_i \times Q_j$ has degree 4.

Let Q_2, Q_3 be such a pair. Consider the largest set of disjoint pairs in $Q_2 \times Q_3$ with one pair of degree 4 and the remaining of degree 3. Note that if there are two disjoint pairs of degree 3 besides the degree 4 pair in the selected set, then clearly we have an H_{432}. So we consider the following two subcases based on whether or not there is a pair of degree 3 in the selected set. Let (b_1, c_1) be the degree 4 pair in the selected set.

Subcase 2.1 There is another pair in $Q_2 \times Q_3$ disjoint from (b_1, c_1) with degree 3.

Assume that $deg(b_2, c_2) = 3$. First observe that

1. If any pair in $\{b_3, b_4\} \times \{c_3, c_4\}$ has degree at least 2 then that pair together with (b_1, c_1) and (b_2, c_2) makes an H_{432}.

2. If both $deg(b_1, c_2)$ and $deg(b_2, c_1)$ are 4 then we get two disjoint degree 4 pairs. Therefore we have that the number of edges incident to pairs in $\{b_1, b_2\} \times \{c_1, c_2\}$ is at most $4+3+4+3 = 14$.

3. If there is a pair in $\{b_3, b_4\} \times \{c_3, c_4\}$ (say (b_3, c_3)) such that the *crossing degree sum* of (b_2, c_2) and (b_3, c_3) is at least 5. Then we must have that one crossing pair is of degree at least 3, and the other is of degree at least 2 (because none of them can be of degree 4). These 2 crossing pairs together with (b_1, c_1) makes the disjoint pairs of an H_{432}. Therefore we must have that (b_2, c_2) and any pair in $\{b_3, b_4\} \times \{c_3, c_4\}$ have their *crossing degree sum* at most 4.

4. Similarly (b_1, c_1) and any pair in $\{b_3, b_4\} \times \{c_3, c_4\}$ have their *crossing degree sum* at most 6.

Assume that (b_1, c_1) and (b_3, c_3) have their crossing degree sum, equal to 6. If the degrees of crossing pairs are 4 and 2, then these crossing pairs and (b_2, c_2) makes the disjoint pairs of an H_{432}. On the other hand if both the crossing pairs have degree 3. Then (b_1, c_3), (b_3, c_1) and (b_2, c_2) are three disjoint pairs of degree 3. From observation 1 we have $deg(b_3, c_3) \leq 1$ and from observation 3 the crossing degree sum of (b_3, c_3) and (b_2, c_2) is at most 4. Which together with observation 2 gives us that the total number of edges incident to pairs in $\{b_1, b_2, b_3\} \times \{c_1, c_2, c_3\}$ is at most $14 + 6 + 1 + 4 = 25$. Now if $deg(b_4, c_4) = 1$ then we have an H_{3321}, otherwise we have that the number of edges containing either b_4 or c_4 is at least $37 - 25 \geq 12$. By observation 1 we have that both $deg(b_4, c_3)$ and $deg(b_3, c_4)$ are at most 1 hence we must have that the *crossing degree sum* of (b_2, c_2) and (b_4, c_4) is 4 with one crossing pair of degree at least 1 and the other of degree at least 2. These crossing pairs together with (b_1, c_3) and (b_3, c_1) gives us an H_{3321}.

On the other hand if for any pair in $\{b_3, b_4\} \times \{c_3, c_4\}$ and (b_1, c_1) their *crossing degree sum* is be at most 5. Then the number of edges incident to pairs in $\{b_3, b_4\} \times \{c_3, c_4\}$ is at least $37 - 14 - 4(2) - 5(2) = 5$. Which implies that there must a degree 2 pair, and hence by observation 1, we get an H_{432}.

Subcase 2.2 There is no pair of degree 3 disjoint from (b_1, c_1).

In this case again as in observation 2, the *crossing degree sum* of (b_1, c_1) and any pair in $\{b_2, b_3, b_4\} \times \{c_2, c_3, c_4\}$ is at most 6 (as any other case results in two disjoint pairs of degree 4 and 3). This implies that the number of edges incident to pairs in $\{b_2, b_3, b_4\} \times \{c_2, c_3, c_4\}$ is at least $37 - 4 - 3(6) = 15$ and no pair has degree 3. Which implies that there are three disjoint pairs in $\{b_2, b_3, b_4\} \times \{c_2, c_3, c_4\}$ with degrees 2, 2 and at least 1 respectively. These pairs and (b_1, c_1) makes the 4 disjoint pairs of an H_{4221}.

Case 3: In every Q_i and Q_j, $i \neq j$ there are exactly two disjoint pairs in $Q_i \times Q_j$ with degree 4.

11

Consider Q_1, Q_2 and assume that (a_1, b_1) and (a_2, b_2) are the two disjoint pairs with degree 4. We make the following observations:

1. If any pair in $\{a_3, a_4\} \times \{b_3, b_4\}$ has degree at least 2 then that pair together with (a_1, b_1) and (a_2, b_2) makes the disjoint pairs of an H_{432}. Therefore the total number of edges spanned by pairs in $\{a_3, a_4\} \times \{b_3, b_4\}$ is at most 4.

2. For any of (a_1, b_1) and (a_2, b_2) and any pair in $\{a_3, a_4\} \times \{b_3, b_4\}$ their *crossing degree sum* can be at most 5. Indeed otherwise say the *crossing degree sum* of (a_1, b_1) and (a_3, b_3) is 6, then we must have that one crossing pair has degree at least 3, and the other has degree at least 2. These crossing pairs and (a_2, b_2) make the disjoint pairs of an H_{432}. Furthermore if any such *crossing degree sum* is 5 then by the same reasoning as above, it must be that one crossing pair is of degree 4 and the other is of degree 1.

3. If the total number of edges spanned by pairs in $\{a_1, a_2\} \times \{b_1, b_2\}$ is at most 12, then the number of edges that uses one vertex from $\{a_1, a_2, b_1, b_2\}$ and one vertex from $\{a_3, a_4, b_3, b_4\}$ is at least $37 - 12 - 4 = 21$. Hence there will be a pair in $\{a_3, a_4\} \times \{b_3, b_4\}$ (say (a_3, b_3)) such that *crossing degree sum* of (a_3, b_3) and at least one of (a_1, b_1) and (a_2, b_2) is at least 6, and by observation 2 we get an H_{432}.

The above observations are true for any two disjoint pairs of degree 4. We choose two disjoint pairs of degree 4, (say (a_1, b_1) and (a_2, b_2)) pairs in $Q_1 \times Q_2$ such that (i) a_1 has the maximum vertex degree among all vertices that are part of some pairs of degree 4 and (ii) $deg(a_2, b_1)$ is as small as possible. Let $deg(a_2, b_1) = x$. Note that by observation 3, we have $x \geq 1$. We consider the following cases based on the value of x.

Subcase 3.1 $x = 4$

Note that that the number of edges in $\{a_1, a_2\} \times \{b_1, b_2\}$ is at most 16 ($deg(a_1, b_2) \leq 4$). First we will show that both $deg(a_3, b_1), deg(a_4, b_1) \leq 1$. Assume that $deg(a_3, b_1) \geq 2$, but then as in observation 2 both $deg(a_1, b_3)$ and $deg(a_1, b_4)$ can be at most 2. Hence by the maximality of the degree of a_1 we get $deg(a_3, b_1) + deg(a_4, b_1) \leq 4$. Now by observation 1, there must be at least $37 - 16 - 4 - 8 = 9$ edges containing one vertex from $\{a_3, a_4, b_3, b_4\}$ and one of $\{a_2, b_2\}$. Which implies that the degree of a_2 or b_2 is strictly larger than that of a_1, a contradiction. So we have $deg(a_3, b_1) = deg(a_4, b_1) = 1$.

Now we show that both $deg(a_3, b_2), deg(a_4, b_2) \leq 1$. To see this first assume that either $deg(a_1, b_3)$ or $deg(a_1, b_4)$ is equal to 4, (say $deg(a_1, b_3) = 4$) then by the minimality of x we must have that $deg(a_2, b_3) = 4$ too, because if $deg(a_2, b_3) < x$, then we can exchange b_1 with b_3 to get a smaller

value of x. But if $deg(a_2, b_3) = 4$ then by observation 2 we must have $deg(a_3, b_2) = deg(a_4, b_2) = 1$. On the other hand if both $deg(a_1, b_3)$ and $deg(a_1, b_4)$ are at most 3 $(deg(a_1, b_3) + deg(a_1, b_4) \leq 6)$ then again there must be at least $37 - 16 - 4 - 8 = 9$ edges containing some pair in $\{a_2\} \times \{b_3, b_4\}$ and $\{a_3, a_4\} \times \{b_2\}$. Which means at least one of these pairs must be of degree at least 3. Say $deg(a_2, b_3) \geq 3$, but then by observation 2 we have $deg(a_3, b_2), deg(a_4, b_2) \leq 1$ and we are done. In case say $deg(a_3, b_2) \geq 3$ then we have $deg(a_3, b_2) + deg(a_4, b_2) \geq 7$ and we get that the degree of b_2 is larger than degree of a_1, a contradiction.

So we have that $deg(a_3, b_1), deg(a_4, b_1), deg(a_3, b_2)$ and $deg(a_4, b_2) \leq 1$. This together with observation 1 implies that the number of edges containing some pair in $\{a_1, a_2\} \times \{b_1, b_2, b_3, b_4\}$ is at least $37 - 4 - 4 = 29$. Therefore the $2 \times 4 \times 4$ 3-graph $H'(\{b_1, b_2\}, Q_2, Q_3)$ has at least $37 - 4 - 4 = 29$ edges. There must be at least three vertices in Q_2 such that each one of them is part of at least three pairs in $(Q_2 \times Q_3)$ that are of degree at least 2. To see this assume that there are at most two such vertices in Q_2 (say b_1 and b_2). Then using the fact that the maximum degree of a pair in $Q_2 \times Q_3$ in H' is 2, we get that b_1 and b_2 can be contained in at most $2(4 \cdot 2) = 16$ edges. While at most 2 pairs containing either b_3 and b_4 can be of degree 2, we get that the number of edges containing either b_3 and b_4 is at most $2 \cdot (2 \cdot 2 + 2 \cdot 1) = 12$ which implies that $|E(H')| \leq 28$ a contradiction

Now since in Q_2 there are at least 3 vertices such that each one of them is part of at least 3 pairs in $(Q_2 \times Q_3)$ of degree at least 2. Which implies that there must a degree 2 pair disjoint from the two degree 4 pairs (guaranteed in Case 3) in $Q_2 \times Q_3$. Therefore we get an H_{432}.

Subcase 3.2 $x = 3$

Now we have that the number of edges in $\{a_1, a_2\} \times \{b_1, b_2\}$ is at most 15. Again we first show that both $deg(a_3, b_1), deg(a_4, b_1) \leq 1$. Assume that $deg(a_3, b_1) \geq 2$, but then by observation 2 both $deg(a_1, b_3)$ and $deg(a_1, b_4)$ can be at most 2. Hence by the maximality of the degree of a_1 we get $deg(a_3, b_1) + deg(a_4, b_1) \leq 5$. Now by observation 1, there must be at least $37 - 15 - 4 - 9 = 9$ edges containing one vertex from $\{a_3, a_4, b_3, b_4\}$ and one of $\{a_2, b_2\}$. Which implies that the degree of a_2 or b_2 is strictly larger than that of a_1, a contradiction. So we have $deg(a_3, b_1) = deg(a_4, b_1) = 1$.

Similarly as in the previous case we show that both $deg(a_3, b_2), deg(a_4, b_2) \leq 1$. To see this first assume that either $deg(a_1, b_3)$ or $deg(a_1, b_4)$ is equal to 4, (say $deg(a_1, b_3) = 4$) then by the minimality of x we must have that $deg(a_2, b_3) \geq 3$ too. But if $deg(a_2, b_3) = 3$ then by observation 2 we must have $deg(a_3, b_2), deg(a_4, b_2) \leq 1$. On the other hand if both $deg(a_1, b_3)$ and $deg(a_1, b_4)$ are at most 3 $(deg(a_1, b_3) + deg(a_1, b_4) \leq 6)$ then again there must be at least $37 - 15 - 4 - 8 = 9$ edges containing some pair in $\{a_2\} \times \{b_3, b_4\}$ and $\{a_3, a_4\} \times \{b_2\}$. Which means at least one of these pairs must be of degree at least 3. Say $deg(a_2, b_3) \geq 3$, but then by observation 2 we have $deg(a_3, b_2), deg(a_4, b_2) \leq 1$

and we are done. In case say $deg(a_3, b_2) \geq 3$ then we have $deg(a_3, b_2) + deg(a_4, b_2) \geq 7$ and we get that the degree of b_2 is greater than the degree of a_1, a contradiction.

So we have that $deg(a_3, b_1), deg(a_4, b_1), deg(a_3, b_2)$ and $deg(a_4, b_2) \leq 1$. Again we get that the $2 \times 4 \times 4$ 3-graph $H'(\{b_1, b_2\}, Q_2, Q_3)$ has at least $37 - 4 - 4 = 29$ edges. and we are done.

Subcase 3.3 $x = 2$

Similarly as in the previous two subcases we have that $deg(a_3, b_1), deg(a_4, b_1), deg(a_3, b_2)$ and $deg(a_4, b_2) \leq 1$ and the $2 \times 4 \times 4$ 3-graph $H'(\{b_1, b_2\}, Q_2, Q_3)$ has at least $37 - 4 - 4 = 29$ edges and we are done.

Subcase 3.4 $x = 1$

In this case observation 3 implies that the number of edges in $\{a_1, a_2\} \times \{b_1, b_2\}$ is exactly 13 $(deg(a_1, b_2) = 4)$. Using $|E(H) \geq 37|$ and observation 2 we get that every pair in $\{a_3, a_4\} \times \{b_3, b_4\}$ has degree exactly 1 and for any pair in $\{a_3, a_4\} \times \{b_3, b_4\}$ and any of (a_1, b_1) or (a_2, b_2) their *crossing degree sum* is exactly 5 $(4 + 1)$.

Therefore, we have that either

$$deg(a_1, b_3) = deg(a_1, b_4) = deg(a_3, b_2) = deg(a_4, b_2) = 4 \quad \text{or}$$

$$deg(a_1, b_3) = deg(a_1, b_4) = deg(a_2, b_3) = deg(a_2, b_4) = 4$$

In the latter case note that again we have that $H'(\{b_1, b_2\}, Q_2, Q_3)$ has at least 29 edges and we are done as above.

So assume that $deg(a_1, b_3) = deg(a_1, b_4) = deg(a_3, b_2) = deg(a_4, b_2) = 4$ and $deg(a_1, b_1) = deg(a_2, b_2) = deg(a_1, b_2) = 4$ and every other pair in $Q_1 \times Q_2$ is of degree exactly 1. This means that a_1 and b_2 are not part of any degree 1 pair. Now the neighborhoods of all the degree 1 pairs must be the same vertex in Q_3 (say c_3). Because otherwise we get a perfect matching in H, (using those two vertices in Q_3 for two disjoint degree 1 pairs, and the remaining two vertices of Q_3 are matched with two of the degree 4 pairs in $(Q_1 \times Q_2)$). But if all of these edges are incident to c_3, then all edges in this graph are incident to at least one vertex in $\{a_1, b_2, c_3\}$ and the number of edges is exactly 37, hence H is isomorphic to H_{ext}. $\square$

Let A, B and C be three disjoint balanced complete 4-partite 4-graphs with color classes $(A_1, \ldots, A_4)$, $(B_1, \ldots, B_4)$ and $(C_1, \ldots, C_4)$ respectively, and $|A_1| = |B_1| = |C_1| = m$. Let Z be a set of vertices disjoint from vertices in A, B and C. For a small constant $\eta > 0$, we say that Z is *connected* to

a triplet of color classes (A_i, B_j, C_k), $1 \leq i, j, k \leq 4$, if $d_4(Z, (A_i \times B_j \times C_k)) \geq 2\eta$. For Z and (A, B, C) we define an auxiliary graph, (the *link graph*), L_{abc} to be a $4 \times 4 \times 4$ 3-graph where the vertex set of each color class of L_{abc} corresponds to the color classes in A, B and C. While a triplet of vertices (a_i, b_j, c_k) is an edge in L_{abc} iff Z is connected to the triplet of color classes (A_i, B_j, C_k).

Given three balanced complete 4-partite 4-graphs A, B and C and another set of vertices Z, as above, we say that we can *extend* (A, B, C) if we can build another set of balanced complete 4-partite 4-graphs using $V(A) \cup V(B) \cup V(C) \cup Z$ such that the total number of vertices in the new 4-partite 4-graphs is at least $12m + \eta m/16$ and the size of a color class in each new 4-partite 4-graph is $\beta \sqrt{\log m}$. In what follows we outline a procedure to extend (A, B, C) using the structure of L_{abc}.

Lemma 12. *For $\eta, c > 0$, let A, B and C be three balanced complete 4-partite 4-graphs such that $|A_1| = |B_1| = |C_1| = m$. If Z is a disjoint set of vertices with $|Z| \geq c2^{m^3}$. If the link graph L_{abc} has at least 37 edges and L_{abc} is not isomorphic to H_{ext} then we can extend (A, B, C).*

Proof. Since L_{abc} is a $4 \times 4 \times 4$ 3-graph with at least 37 edges and is not isomorphic to H_{ext}, for each of the other cases as in Lemma 11, we give the procedure to extend (A, B, C).

Case 1: L_{abc} has a perfect matching:

Without loss of generality assume that the perfect matching in L_{abc} corresponds to $\{(A_i, B_i, C_i) : 1 \leq i \leq 4\}$ i.e. Z is connected to the triplets $\{(A_i, B_i, C_i) : 1 \leq i \leq 4\}$. Note that by the definition of connectedness and the sizes of the sets the 4-partite 4-graph (A_1, B_1, C_1, Z) satisfies the conditions of Lemma 7. Hence we find a complete balanced 4-partite 4-graph $X_1 = (A_1^1, B_1^1, C_1^1, Z^1)$, such that

$$Z^1 \subset Z, \ \ A_1^1 \subset A_1 \ , \ B_1^1 \subset B_1 \text{ and } C_1^1 \subset C_1 \ \text{ and}$$

$$|Z^1| = |A_1^1| = |B_1^1| = |C_1^1| = \beta \sqrt{\log m} \ \text{ where } \beta \text{ is as in Lemma 7}$$

Similarly, we find such complete balanced 4-partite 4-graphs X_2, X_3 and X_4 in (A_2, B_2, C_2, Z), (A_3, B_3, C_3, Z) and (A_4, B_4, C_4, Z) respectively, that are disjoint from each other (as $|Z|$ is very large compared to m) (see Figure 3). We remove the vertices in X_1, X_2, X_3 and X_4 and make these four new 4-partite 4-graphs. In the remaining parts of A, B and C we remove another such set of 4 disjoint complete balanced 4-partite 4-graphs. Again by definition of connectedness and Lemma 7 we can continue this process until we remove at least $\eta m/8$ vertices from each color class of A, B and C.

Note that the new 4-partite 4-graphs use at least $4\eta m/8$ vertices from Z. Therefore these new 4-partite 4-graphs together with leftover parts of A, B and C have at least $3(4m) + \eta m/2$ vertices while all the 4-partite 4-graphs are balanced. Hence, we extended (A, B, C).
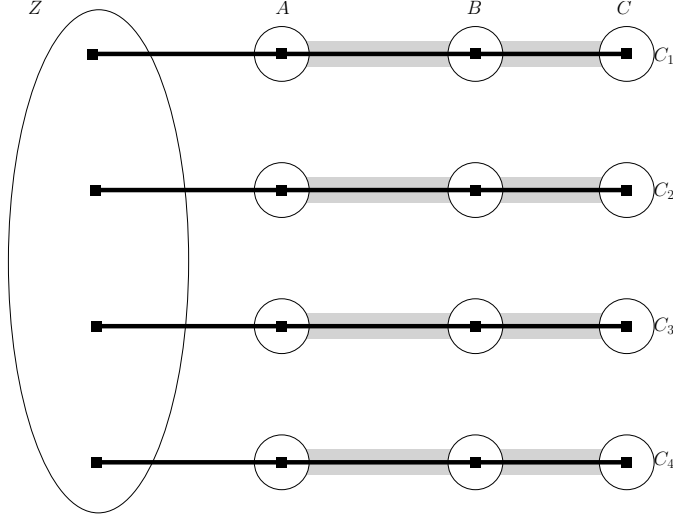
Figure 3: Extending (A, B, C) when L_{abc} has a perfect matching: The shaded boxes represent a triplet *connected* to Z, while solid thick lines represent a balanced complete 4 partite graphs

Case 2: L_{abc} has a subgraph isomorphic to H_{432}:

In this case we show in detail how to extend such an (A, B, C), while in the latter cases we will only briefly outline the procedure. First assume that the H_{432} in L_{abc} is as in Figure 2(a) and let the pairs corresponding to the degree 4, 3 and 2 pairs in this subgraph be (B_1, C_1), (B_2, C_2) and (B_3, C_3) respectively. Furthermore let the color classes corresponding to the neighbors of degree 3 and degree 2 pairs in H_{432} be $\{A_2, A_3, A_4\}$ and $\{A_3, A_4\}$ respectively.

Using the definition of connectedness and Lemma 7 we find two disjoint complete balanced 4-partite 4-graphs $X_1 = (A_4^1, B_3^1, C_3^1, Z^1)$ and $X_2 = (A_3^2, B_3^2, C_3^2, Z^2)$ such that Z^j, A_i^j, B_i^j, C_i^j are subsets of Z, A_i, B_i, C_i respectively.

Similarly we find three more disjoint balanced complete 4-partite 4-graphs X_3, X_4 and X_5 where vertices of three color classes in all of them are from B_2, C_2 and Z while vertices of the fourth color class are from A_2, A_3 and A_4 respectively. We build two more disjoint balanced complete 4-partite 4-graphs X_6 and X_7 such that vertices of three color classes in both of them are from B_1, C_1 and Z while vertices of the fourth color class are from A_1 and A_2 respectively.

The size of a color class in $X_1, \ldots, X_7$ is $\beta\sqrt{\log m}$. We remove the vertices in $X_1, \ldots X_7$ from their color classes to make these 7 new 4-partite 4-graphs

In the remaining parts of A, B and C we remove another such set of seven disjoint balanced complete 4-partite 4-graphs that are disjoint from the previous ones. Again by definition of con-
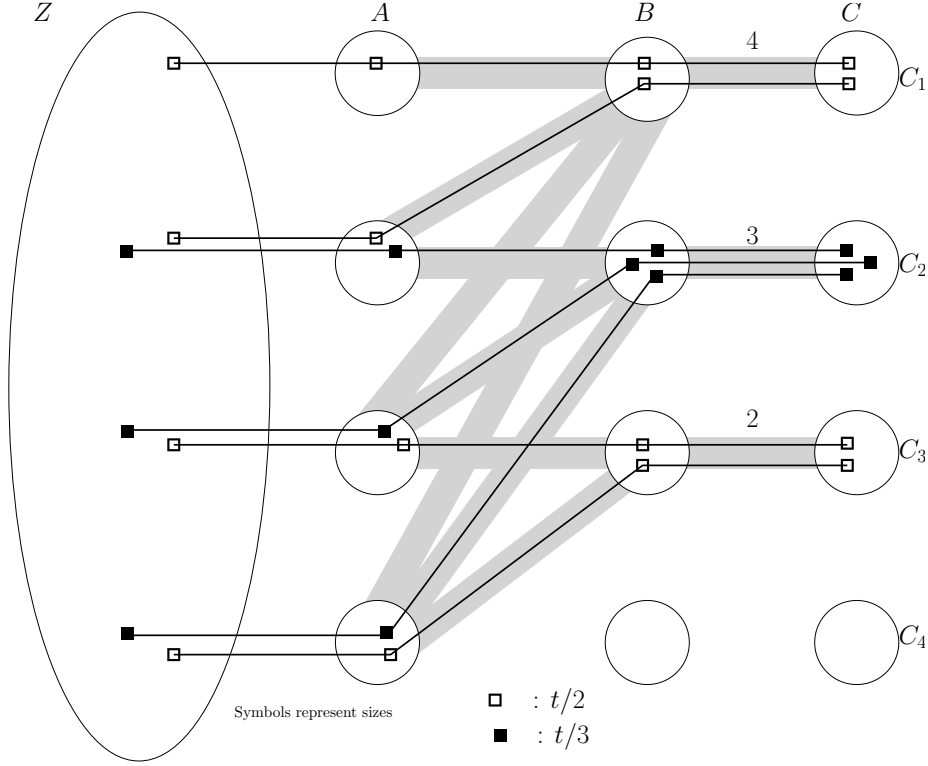
16

Figure 4: Extending (A, B, C) when L_{abc} has an H_{432} as in Figure 2(a): The shaded boxes represent a triplet *connected* to Z, while solid think lines represent a balanced complete 4 partite 4-graphs

nectedness and Lemma 7 we can continue this process until we remove $\eta m/8$ vertices each from B_i and C_i, $1 \leq i \leq 3$. By construction, if the number of vertices used from B_i and C_i, $1 \leq i \leq 3$ is t $(= \eta m/8)$ then the number of vertices used in A_2, A_3 and A_4 is $5t/6$, while that in A_1 is $t/2$ (see Figure 4).

Note that the new 4-partite 4-graphs use at least $3t \geq 3\eta m/8$ vertices from Z, but the remaining parts of A, B and C are not balanced (A_1, B_4 and C_4 have more vertices). To restore the balance in the remaining part of A we discard some arbitrary $t/3$ vertices from the remaining part of A_1. Similarly we discard some arbitrary t vertices from B_4 and C_4 to restore the balance in the remaining part of B and C. Therefore the new 4-partite 4-graphs together with leftover parts of A, B and C (after discarding the vertices) have at least $4(|A_1| + |B_1| + |C_1|) + 3t - t/3 - 2t \geq 12m + \eta m/12$ vertices while all the 4-partite 4-graphs are balanced. Hence we extended (A, B, C).

On the other hand if the H_{432} in L_{abc} is as in Figure 2(b), then let the color classes corresponding to the neighbors of degree 3 and degree 2 pairs in H_{432} be $\{A_1, A_2, A_3\}$ and $\{A_3, A_4\}$ respectively.

Figure 5: Extending (A, B, C) when L_{abc} has an H_{432} as in Figure 2(b): The shaded boxes represent a triplet *connected* to Z, while solid thick lines represent a balanced complete 4 partite 4-graphs

In this case extend (A, B, C) as follows.

For the pair (B_3, C_3) we remove two balanced complete 4-partite 4-graphs with the fourth color classes in A_3 and A_4 respectively. For the pair B_2, C_2 we remove two balanced complete 4-partite 4-graphs with the fourth color classes in A_1 and A_2 respectively. The size of of each color class in all of these new 4-partite graphs is $\beta\sqrt{\log m}$. Since the pair (B_1, C_1) has degree 4, we remove four balanced complete 4-partite 4-graphs with the fourth color class in A_1, A_2, A_3 and A_4 respectively.

Similarly as in the previous case we repeat this process so that we remove at least $t \geq \eta m/8$ vertices from each B_i and C_i, $1 \leq i \leq 3$. Note that by construction we have used $3t/4$ vertices in each color class of A (see Figure 5). So the remaining part of A is still balanced. While to restore balance in the remaining parts of B and C, we discard some arbitrary t vertices from each of B_4 and C_4. Again in total we added $3t$ vertices from Z, while we discarded $2t$ vertices form B_4 and C_4. Therefore the net increase in the number of vertices in the new set of complete 4-partite 4-graphs is $t \geq \eta m/8$, while all the 4-partite 4-graphs are balanced.

18

Case 3: L_{abc} has a subgraph isomorphic to H_{4221}:

Without loss of generality, assume that the pairs corresponding to the degree 4, 2, 2 and 1 pairs in this H_{4221} are (B_1, C_1), (B_2, C_2), (B_3, C_3) and (B_4, C_4) respectively. Furthermore let the color classes corresponding to the neighbors of degree 1 and the two degree 2 pairs in H_{4221} be $\{A_4\}$, $\{A_3, A_4\}$ and $\{A_3, A_4\}$ respectively (as in Figure 3(a)). By the definition of connectedness and Lemma 7 we build complete balanced 4-partite 4-graphs using (A_4, B_4, C_4, Z) and (A_3, B_3, C_3, Z) of size $\beta\sqrt{\log m}$. For the pair (B_2, C_2) we make two more balanced complete 4-partite 4-graphs using (A_4, B_2, C_2, Z) and (A_3, B_2, C_2, Z). For the degree 4 pair (B_1, C_1) we remove balanced complete 4-partite 4-graphs in (A_1, B_1, C_1, Z) and (A_2, B_1, C_1, Z). Again we repeat this process so that we
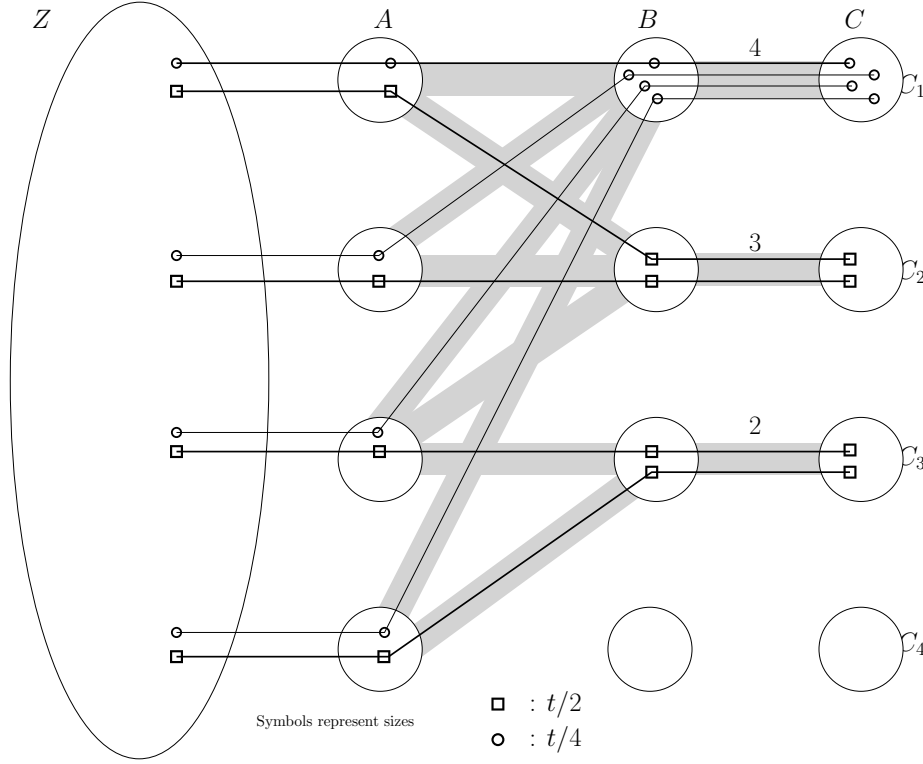

Figure 6: Extending (A, B, C) when L_{abc} has an H_{4221}: The shaded boxes represent a triplet *connected* to Z, while solid thick lines represent a balanced complete 4 partite 4-graphs

remove $t \geq \eta m/24$ vertices from each color class of B and C. Note that with this process we have used $3t/2$ vertices in A_3 and A_4 while $t/2$ vertices each in A_1 and A_2 (see Figure 6). Furthermore only the remaining part of A is not balanced. The balance can be restored by discarding t vertices each from the remaining part of A_1 and A_2 which results in the net increase of $2t \geq \eta m/12$ vertices in all the balanced complete tripartite graphs.

Case 4: L_{abc} has a subgraph isomorphic to H_{3321}:

Assume that the pairs corresponding to the degree 3, 3, 2 and 1 pairs in the H_{3321} are (B_1, C_1), (B_2, C_2), (B_3, C_3) and (B_4, C_4) respectively. Let the color classes corresponding to the neighbors of these pairs in H_{3321} be $\{A_2, A_3, A_4\}$, $\{A_2, A_3, A_4\}$, $\{A_3, A_4\}$ and $\{A_4\}$. By the definition of connectedness and Lemma 7 we build three complete balanced 4-partite graphs in each of (A_4, B_4, C_4, Z), (A_3, B_3, C_3, Z) and (A_2, B_2, C_2, Z) of size $\beta\sqrt{\log m}$. In addition we build three more complete balanced 4-partite 4-graphs using (B_1, C_1) and Z, while the vertices of the fourth color classes are in A_2, A_3 and A_4 respectively. We repeat this process so as to remove at least $t \geq 3\eta m/64$ vertices each from each color class of B and C. Clearly the remaining part of B and C are still balanced, while in A we have used $4t/3$ vertices in each of A_2, A_3 and A_4. To restore the balance in remaining part of A we discard arbitrary $4t/3$ vertices from A_1. In the process the net increase in the number of vertices in the resultant balanced 4-partite 4-graphs is at least $8t/3 \geq 3\eta m/8$, hence (A, B, C) is *extended*. □

Figure 7: Extending (A, B, C) when L_{abc} has an H_{3321}: The shaded boxes represent a triplet *connected* to Z, while solid thick lines represent a balanced complete 4 partite 4-graphs

4   The Non Extremal Case

Throughout this section we assume that we have a 4-graph H satisfying (1) such that the extremal case does not hold for H. We shall assume that n is sufficiently large and besides our main parameter γ we use the parameters β and α such that the following holds

$$\frac{12}{2^{1/\beta^2}} < \gamma = \alpha^4 \ll 1 \tag{2}$$

where $a \ll b$ means that a is sufficiently small compared to b. From (1) and (2), when n is large we have

$$\delta_1(H) \geq \binom{n-1}{3} - \binom{3n/4}{3} + 1 > \frac{37}{64}\binom{n-1}{3} - \frac{9n^2}{64} > \left(1/2 + 2\sqrt{\alpha}\right)\binom{n}{3}$$

Hence our hypergraph H satisfies the conditions of Lemma 10 (the absorbing lemma). We remove from H an *absorbing matching* M of size at most $\alpha^2 n/4 = \sqrt{\gamma}n/4$. In the remaining hypergraph we find an almost perfect matching that leaves out a set of at most $\alpha^4 n = \gamma n$ vertices. As guaranteed by Lemma 10 the vertices that are left out from the almost perfect matching can be absorbed into M, therefore we get a perfect matching in H. In what follows we work with the remaining hypergraph (after removing $V(M)$). For simplicity we still denote the remaining hypergraph by H and assume that it is on n vertices. Since $|V(M)| \leq \sqrt{\gamma}n$, in the remaining hypergraph we still have

$$\delta_1(H) \geq \left(\frac{37}{64} - 6\sqrt{\gamma}\right)\binom{n}{3} \tag{3}$$

as for any vertex v in the remaining hypergraph, there can be at most $6\sqrt{\gamma}\binom{n}{3}$ edges containing v and at least one vertex in $V(M)$.

4.1   The optimal cover

Our goal is to find an almost perfect matching in H. We are going to build a cover $\mathcal{F} = \{Q_1, Q_2, \ldots\}$, such that, each Q_i is a disjoint balanced complete 4-partite 4-graph in H (we refer to them as 4-partite graphs). We say that such a cover is *optimal* if it covers at least $(1 - \gamma)n$ vertices. We will show that either we can find an optimal cover or H is α-extremal. It is easy to see that such an optimal cover readily gives us an almost perfect matching.

Using the following iterative procedure, we either build an optimal cover or find a subset of vertices, which shows that H is α-extremal. We begin with a cover $\mathcal{F}_0$. Then in each step $t \geq 1$, if $\mathcal{F}_{t-1}$ is not optimal, we find another cover $\mathcal{F}_t$, such that $|V(\mathcal{F}_t)| \geq |V(\mathcal{F}_{t-1})| + \gamma^2 n/16$ (for this we use the notation, $\mathcal{F}_t > \mathcal{F}_{t-1}$). The size of a color class in each 4-partite graph in $\mathcal{F}_t$ is m_t.

To get the initial cover, $\mathcal{F}_0$, we repeatedly apply Lemma 4 in the leftover of H, while the conditions of the lemma are satisfied and the number of leftover vertices are at least γn, to find 4-partite graphs, $K^{(4)}(m_0)$, where $m_0 = \beta (\log n)^{1/3}$. After the tth step in this iterative procedure, if $\mathcal{F}_t$ is not an optimal cover, then we get $\mathcal{F}_{t+1}$. We will show that, unless H is α-extremal, we have $\mathcal{F}_{t+1} > \mathcal{F}_t$ and $m_{t+1} = \beta \sqrt{\log m_t}$. Let $\mathcal{I}_t = V(H) \setminus V(\mathcal{F}_t)$. Since $\mathcal{F}_t$ is not optimal and we cannot apply Lemma 4 in $H|_{\mathcal{I}_t}$, we must have that $|\mathcal{I}_t| \geq \gamma n$ and

$$d_4(\mathcal{I}_t) < \gamma. \tag{4}$$

By non-extremality of H, this implies that $|V(\mathcal{F}_0)| \geq n/4$.

In what follows we will show that if there are 'many' edges with three vertices in $\mathcal{I}_t$ and one vertex in some $Q_i \in \mathcal{F}_t$ then we get $\mathcal{F}_{t+1} > \mathcal{F}_t$. To that end let $Q_i = (V_1^i, V_2^i, V_3^i, V_4^i)$ be a 4-partite graph in $\mathcal{F}_t$, we say that a color class, V_p^i of Q_i, $(1 \leq p \leq 4)$ is *connected* to $\mathcal{I}_t$, if $d_4(V_p^i, \binom{\mathcal{I}_t}{3}) \geq 2\gamma$. We will show that if a 4γ-fraction of the 4-partite graphs in $\mathcal{F}_t$ have at least 2 color classes connected to $\mathcal{I}_t$, then we can we get $\mathcal{F}_{t+1} > \mathcal{F}_t$. To see this assume that we have a subcover $\mathcal{F}' \subset \mathcal{F}_t$, such that each $Q_i \in \mathcal{F}'$ has at least 2 color classes connected to $\mathcal{I}_t$ and $|V(\mathcal{F}')| \geq \gamma n$. Let $\mathcal{F}' = \{Q_1, Q_2, \ldots\} \subset \mathcal{F}_t$ and without loss of generality, say V_1^i and V_2^i are the color classes in each Q_i that are connected to $\mathcal{I}_t$. For each such Q_i, since $|V_1^i| = |V_2^i| = m_t \leq \beta (\log n)^{1/3}$ and $|\mathcal{I}_t| \geq \gamma n$, by Lemma 5 we can find two disjoint balanced complete 4-partite graphs $(U_1^i, A_1^i, B_1^i, C_1^i)$ and $(U_2^i, A_2^i, B_2^i, C_2^i)$ where U_1^i and U_2^i are subsets of V_1^i and V_2^i respectively, and A_k^i, B_k^i and C_k^i, $k \in \{1, 2\}$ are disjoint subsets of $\mathcal{I}_t$. The size of each color class of these new 4-partite graphs is at least $\gamma m_t/4$ (see Figure 8).
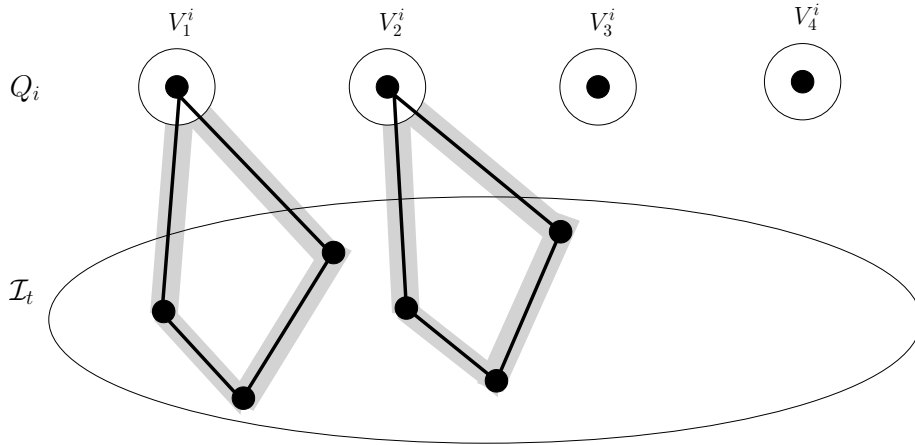

Figure 8: V_1^i and V_2^i of Q_i are connected to $\mathcal{I}_t$. The Shaded quadrilaterals represent density, while the dark black quadrilateral is balanced complete 4-partite 4-graph

We remove the vertices of these new 4-partite graphs from their respective sets and add them

to $\mathcal{F}_{t+1}$. Removing these vertices from V_1^i and V_2^i creates an imbalance in the leftover part of Q_i (V_3^i and V_4^i have more vertices). To restore the balance, we discard (add to $\mathcal{I}_t$) some arbitrary $|U_1^i| = |U_2^i|$ vertices each from V_3^i and V_4^i. The new 4-partite graphs use $6\gamma m_t/4$ vertices from $\mathcal{I}_t$. Therefore even after discarding the vertices from V_3^i and V_4^i the net increase in the size of our cover is γm_t, while all the 4-partite graphs are balanced. We repeat this procedure for every $Q_i \in \mathcal{F}'$ and add the leftover part of Q_i and all 4-partite graphs in $\mathcal{F}_t \setminus \mathcal{F}'$, to $\mathcal{F}_{t+1}$. Now, we split each 4-partite graph in $\mathcal{F}_{t+1}$ into disjoint balanced complete 4-partite graphs, such that each has a color class of size $m_{t+1} = \beta\sqrt{\log m_t}$ (we assume divisibility). Since $|V(\mathcal{F}')| \geq \gamma n$, by the above observation, we have $|V(\mathcal{F}_{t+1})| \geq |V(\mathcal{F}_t)| + \gamma^2 n$, hence $\mathcal{F}_{t+1} > \mathcal{F}_t$.

Similarly, if there are 'many' edges that uses two vertices in $\mathcal{I}_t$ and two vertices from $V(\mathcal{F}_t)$ then we get $\mathcal{F}_{t+1} > \mathcal{F}_t$. First note that the number of pairs of vertices of any 4-partite graph $Q_i \in \mathcal{F}_t$ is $O(\log n)^{2/3}$, so the number of pairs of vertices within the 4-partite graphs in $\mathcal{F}_t$ is $O(n(\log n)^{2/3}) = o\binom{n}{2}$. Therefore the total number of edges, containing two vertices within a $Q_i \in \mathcal{F}_t$ and two vertices in $\mathcal{I}_t$ is $o\binom{n}{4}$, hence we ignore such edges. Let $Q_i = (V_1^i, V_2^i, V_3^i, V_4^i)$ and $Q_j = (V_1^j, V_2^j, V_3^j, V_4^j)$ be a pair of 4-partite graphs in $\mathcal{F}_t$ we say that $\mathcal{I}_t$ is *connected* to a pair of color classes (V_p^i, V_q^j), $(1 \leq p, q \leq 4)$, if $d_4\left(V_p^i, V_q^j, \binom{\mathcal{I}_t}{2}\right) \geq 2\gamma$. We say that $\mathcal{I}_t$ is *k-sided* to a pair $(Q_i, Q_j) \in \binom{\mathcal{F}_t}{2}$ if $\mathcal{I}_t$ is connected to k-pairs in $\{V_1^i, V_2^i, V_3^i, V_4^i\} \times \{V_1^j, V_2^j, V_3^j, V_4^j\}$. Assume that $\mathcal{I}_t$ is at least 9-sided to a 4γ-fraction of pairs of 4-partite graphs in $\binom{\mathcal{F}_t}{2}$. By a simple greedy procedure, (Lemma 8 and the 2-graph analog of Lemma 9) we get a disjoint set of pairs, $M' \subset \binom{\mathcal{F}_t}{2}$, such that for each pair $(Q_i, Q_j) \in M'$, $\mathcal{I}_t$ is at least 9-sided to (Q_i, Q_j) and the number of vertices covered by M' is at least γn.

For every $(Q_i, Q_j) \in M'$ we proceed as follows: Since $\mathcal{I}_t$ is connected to at least 9 pairs of color classes in $\{V_1^i, V_2^i, V_3^i, V_4^i\} \times \{V_1^j, V_2^j, V_3^j, V_4^j\}$, it is easy to see that we can find 3 disjoint pairs of color classes such that $\mathcal{I}_t$ is connected to each of them (say (V_1^i, V_1^j), (V_2^i, V_2^j) and (V_3^i, V_3^j) are such disjoint pairs of color classes). We have $|V_1^i| = |V_1^j| = m_t \leq \beta(\log n)^{1/3}$ and $|\mathcal{I}_t| \geq \gamma n$, so by definition of connectedness, the induced hypergraph, $H\left(V_1^i, V_1^j, \binom{\mathcal{I}_t}{2}\right)$ satisfies the conditions of Lemma 6. Therefore by Lemma 6 we remove balanced complete 4-partite 4-graphs, such that each of them has one color class in V_1^i, one in V_1^j and two color classes in $\mathcal{I}_t$. Note that the conditions of Lemma 6 are satisfied until in total we remove at least $\gamma m_t/2$ vertices each from V_1^i and V_1^j. We repeat the same process with (V_2^i, V_2^j) and (V_3^i, V_3^j) (see Figure 9). In total these new balanced complete 4-partite 4-graphs use $3\gamma m_t$ vertices from $\mathcal{I}_t$ and when we remove the vertices of the new 4-partite graphs, from Q_i and Q_j the remaining parts of Q_i and Q_j are not balanced (V_4^i and V_4^j have extra vertices). To restore the balance we remove an arbitrary set of vertices each from V_4^i and V_4^j (equal to the difference in color classes). Still the net increase in the number of vertices in the cover is at least $2\gamma m_t$.
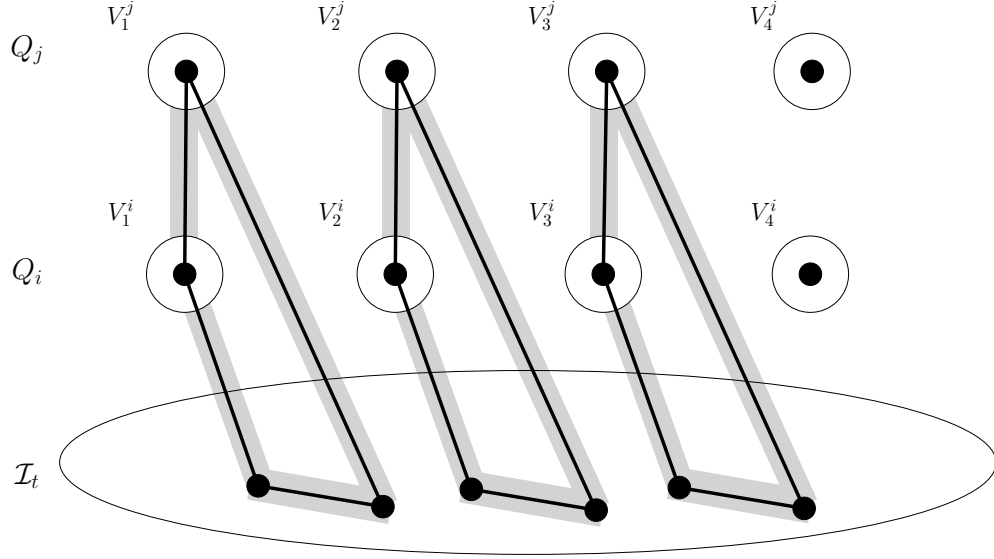
Figure 9: The pair (Q_i, Q_j) is at least 9-sided to $\mathcal{I}_t$. The Shaded quadrilaterals represent density, while the dark black quadrilateral is balanced complete 4-partite 4-graph

Applying this procedure for every $(Q_i, Q_j) \in M'$ and making all of the 4-partite graphs of the same size, by splitting, we get $\mathcal{F}_{t+1}$. Again we have all 4-partite graphs balanced with color classes of size $m_{t+1} = \beta\sqrt{\log m_t}$ and $\mathcal{F}_{t+1} > \mathcal{F}_t$.

Now, if we can not get $\mathcal{F}_{t+1} > \mathcal{F}_t$, we must have that, in almost all 4-partite graphs in $\mathcal{F}_t$, at most one color class is connected to $\mathcal{I}_t$ and almost all pairs of 4-partite graphs in $\binom{\mathcal{F}_t}{2}$ are at most 8-sided. In particular this implies that for a typical vertex $v \in \mathcal{I}_t$ we have

$$deg_4\left(v, \binom{\mathcal{I}_t}{2} \times V\left(\mathcal{F}_t\right)\right) \leq \left(\frac{1}{4} + 6\gamma\right) |V\left(\mathcal{F}_t\right)| \binom{|\mathcal{I}_t|}{2}$$

and

$$deg_4\left(v, \mathcal{I}_t \times \binom{V\left(\mathcal{F}_t\right)}{2}\right) \leq \left(\frac{1}{2} + 16\gamma\right) |\mathcal{I}_t| \binom{|V\left(\mathcal{F}_t\right)|}{2}.$$

24

From (3), (4) and the above degree bounds, for a typical vertex $v \in \mathcal{I}_t$ we have

$$deg_4\left(v, \binom{V(\mathcal{F}_t)}{3}\right)$$

$$\geq \left(\frac{37}{64} - 6\sqrt{\gamma}\right)\binom{n}{3} - d_4\left(v, \mathcal{I}_t \times \binom{V(\mathcal{F}_t)}{2}\right) - d_4\left(v, \binom{\mathcal{I}_t}{2} \times V(\mathcal{F}_t)\right) - d_4\left(v, \binom{\mathcal{I}_t}{3}\right)$$

$$\geq \left(\frac{37}{64} - 6\sqrt{\gamma}\right)\binom{n}{3} - \left(\frac{1}{2} + 16\gamma\right)|\mathcal{I}_t|\binom{|V(\mathcal{F}_t)|}{2} - \left(\frac{1}{4} + 6\gamma\right)\binom{|\mathcal{I}_t|}{2}|V(\mathcal{F}_t)| - \gamma\binom{|\mathcal{I}_t|}{3}$$

$$\geq \left(\frac{37}{64} - 30\sqrt{\gamma}\right)\binom{|V(\mathcal{F}_t)|}{3}$$

where the last inequality holds when $|\mathcal{I}_t| \geq \gamma n$ and $|V(\mathcal{F}_t)| \geq n/4$.

For a vertex v, consider the edges that v makes with 3-sets of vertices within a $Q_i \in \mathcal{F}_t$. The number of 3-sets of vertices of any $Q_i \in \mathcal{F}_{\sqcup}$ is $O(\log n)$, (as the size of Q_i is at most $\beta \log^{1/3} n$), hence the total number of 3-sets of vertices within the 4-partite graphs in $\mathcal{F}_t$ is $O(n \log n) = o\binom{n}{3}$. Similarly the number of 3-sets of vertices which uses 2 vertices from a Q_i and one vertex from some other Q_j is $O(n^2 \log n) = o\binom{n}{3}$. Therefore, for any vertex v, we ignore these types of edges and we will only consider the edges that v makes with 3-sets of vertices (x, y, z), $x \in V(Q_i)$, $y \in V(Q_j)$, $z \in V(Q_k)$, $i \neq j \neq k$. By the above observation, for the minimum degree of a typical vertex $v \in \mathcal{I}_t$ we still have

$$deg_4\left(v, \binom{V(\mathcal{F}_t)}{3}\right) \geq \left(\frac{37}{64} - 40\sqrt{\gamma}\right)\binom{|V(\mathcal{F}_t)|}{3} \tag{5}$$

Let $Q_i = (V_1^i, \ldots, V_4^i)$, $Q_j = (V_1^j, \ldots, V_4^j)$ and $Q_k = (V_1^k, \ldots, V_4^k)$ be three 4-partite graphs in $\mathcal{F}_t$, we say that $\mathcal{I}_t$ is *connected* to a triplet of color classes (V_p^i, V_q^j, V_r^k), $1 \leq p, q, r \leq 4$, if $d_4(\mathcal{I}_t, (V_p^i \times V_q^j \times V_r^k)) \geq 2\gamma$. For $(Q_i, Q_j, Q_k) \in \binom{\mathcal{F}_t}{3}$ we consider the link graph L_{ijk} as defined above (with $\mathcal{I}_t$ playing the role of the set Z).

For a constant $\eta > 0$, we say that $\mathcal{I}_t$ is (η, s)-*connected* to $\mathcal{F}_t$, if there is a subset of triplets of 4-partite graphs, $T \subset \binom{\mathcal{F}_t}{3}$, such that for each triplet $(Q_i, Q_j, Q_k) \in T$, the link graph, L_{ijk} has s edges and $|T| \geq \eta\binom{|\mathcal{F}_t|}{3}$.

A simple calculation, using (5), implies that if $\mathcal{I}_t$ is $(\gamma^{1/3}, s)$-*connected* to $\mathcal{F}_t$ for some $s \leq 36$, then we also have that $\mathcal{I}_t$ is $(\sqrt{\gamma}, \geq 38)$-*connected* to $\mathcal{F}_t$.

We consider the following cases based on the way $\mathcal{I}_t$ is connected to $\mathcal{F}_t$ and show that either we get $\mathcal{F}_{t+1} > \mathcal{F}_t$ or H is extremal. First assume that $\mathcal{I}_t$ is $(32\gamma, \geq 37)$-*connected* to $\mathcal{F}_t$ such that for every triplet $(Q_i, Q_j, Q_k) \in T$, the link graph L_{ijk} is not isomorphic to H_{ext}. Then by lemma 8

and lemma 9 there exists a set a disjoint set of triplets of 4-partite graphs, $T' \subset T$, such that for each triplet $(Q_i, Q_j, Q_k) \in T'$ the link graph L_{ijk} has at least 37 edges and is not isomorphic to H_{ext}. Furthermore, the number of vertices covered by T' is at least γn.

Now, for each $(Q_i, Q_j, Q_k) \in T'$, since L_{ijk} has at least 37 edges and $L_{ijk} \neq H_{ext}$, using Lemma 12 we extend (Q_i, Q_j, Q_k) to add at least $\gamma m_t/16$ vertices to our cover. Clearly if we extend every triplet in T' the net increase in the size of our cover is at least $\gamma^2 n/16$ (as the size of T' is at least γn). Similarly as above we can split the 4-partite graphs to make them of the same size and get $\mathcal{F}_{t+1} > \mathcal{F}_t$.

On the other hand, if there is no such T, then we must have that $\mathcal{I}_t$ is not $(\gamma^{1/3}, s)$-*connected* to $\mathcal{F}_t$ for any $s \leq 36$, because otherwise, as observed above, we will get such a T. So, roughly speaking, for almost all triplets of 4-partite graphs in $\binom{\mathcal{F}_t}{3}$, we have that the link graph of the triplet has exactly 37 edges and is isomorphic to H_{ext}. Call a 4-partite graph $Q_i \in \mathcal{F}_t$ *good*, if for almost all pairs of other 4-partite graphs Q_j, Q_k, we have that the link graph L_{ijk} is isomorphic to H_{ext}.

By the above observation, almost all 4-partite graphs (covering $\geq (1 - 2\gamma^{1/3})|V(\mathcal{F}_t)|$ vertices) are good. By a simple greedy procedure we find a set of disjoint triplets of 4-partite graphs, T_g, such that for each triplet $(Q_i, Q_j, Q_k) \in T_g$, the link graph L_{ijk} is isomorphic to H_{ext} and all good 4-partite graphs are part of some triplet in T_g. Let the set of 4-partite graphs covered by T_g be $\mathcal{F}_g$, clearly $|V(\mathcal{F}_g)| \geq (1 - 2\gamma^{1/3})|V(\mathcal{F}_t)|$. With relabeling we may also assume that in each triplet $(Q_i, Q_j, Q_k) \in T_g$, V_1^i, V_1^j and V_1^k are the color classes corresponding to the vertices of the link graph L_{ijk} that intersect every edge of L_{ijk}.

For every $(Q_i, Q_j, Q_k) \in T_g$, by definition of connectedness and the sizes of the Q_i, Q_j and Q_k, the 4-partite hypergraph induced by $(V_1^i, V_2^j, V_2^k, \mathcal{I}_t)$ satisfies the conditions of Lemma 7. Hence applying Lemma 7 we find a balanced complete 4-partite graphs X_{i1} in $H(V_1^i, V_2^j, V_2^k, \mathcal{I}_t)$. We also find two more disjoint balanced complete 4-partite graphs, X_{i2} and X_{i3} in $H(V_1^i, V_3^j, V_3^k, \mathcal{I}_t)$ and $H(V_1^i, V_4^j, V_4^k, \mathcal{I}_t)$. Since $|\mathcal{I}_t| \geq \gamma n$, we can find these complete 4-partite graphs that are disjoint from each other. The sizes of a color class in each of X_{11}, X_{12} and X_{13} is $\beta\sqrt{\log m_t}$.

Similarly for each of V_1^j and V_1^k we find 3 disjoint balanced complete 4-partite graphs X_{jp} and X_{kp} in $H(V_p^i, V_1^j, V_p^k, \mathcal{I}_t)$ and $H(V_p^i, V_p^j, V_1^k, \mathcal{I}_t)$, $(2 \leq p \leq 4)$, respectively. All of these these balanced complete 4-partite graphs are disjoint from each other and the size of a color class in each one of them is $\beta\sqrt{\log m_t}$.

By the definition of connectedness, the structure of the link graph L_{ijk} and the fact that $|\mathcal{I}_t| \geq \gamma n$, clearly we can find these nine disjoint 4-partite graphs. And as argued above we repeat this process (remove another set of 9 such 4-partite graphs) until in total we remove $\gamma m_t/2$ vertices

from each of V_1^i, V_1^j and V_1^k, while $\gamma m_t/3$ vertices from each of the other classes in Q_i, Q_j and Q_k.

Note that these tripartite graphs in total use $3\gamma m_t/2$ vertices from $\mathcal{I}_t$. But this creates an imbalance among the color classes of the remaining parts of Q_i, Q_j and Q_k (V_1^i, V_1^j and V_1^k have fewer vertices), to restore the balance we will have to discard $\gamma m_t/6$ vertices from each color class in Q_i, Q_j and Q_k except V_1^i, V_1^j and V_1^k. Which leaves us with no net gain in the size of the cover. Therefore we will not discard any vertices from these color classes at this time and say that these color classes have extra vertices. We proceed in similar manner for each triplet in T_g. So we have about $\gamma n/24$ extra vertices altogether.

Denote by V_1^g, V_2^g, V_3^g and V_4^g the union of the corresponding color classes of remaining parts of 4-partite graphs in $\mathcal{F}_g$. Clearly $|V_2^g| = |V_3^g| = |V_4^g| \geq (1-2\gamma^{1/3})|V(\mathcal{F}_t)|/4 - \gamma n/24 \geq (1/16 - 3\gamma^{1/3})n$. The last inequality follows from the lower bound on size of $V(\mathcal{F}_t)$ above, and the fact that γ is a small constant. We will show that either we can increase the size of our cover or we have

$$d_4\left(V_2^g \cup V_3^g \cup V_4^g\right) \leq \sqrt{\gamma}. \tag{6}$$

For $d_4(V_2^g \cup V_3^g \cup V_4^g)$, we only consider those edges that use exactly one vertex from a 4-partite graph Q_i, as the number of edges of other types is $o(n^4)$. Assume that $d_4(V_2^g \cup V_3^g \cup V_4^g) \geq \sqrt{\gamma}$ then by Lemma 4 there exist complete 4-partite graphs in $H|_{V_2^g \cup V_3^g \cup V_4^g}$ covering at least γn vertices. We remove some of these 4-partite graphs (possibly with splitting and discarding part of it) such that from no color class we remove more then the number of extra vertices in that color class. Adding these new 4-partite graphs to our cover increases the size of our cover by at least $\gamma^2 n$ vertices. As we will not need to discard vertices from $(V_2^g \cup V_3^g \cup V_4^g)$ for rebalancing. Instead the extra vertices are part of these new 4-partite graphs. In the remaining parts of V_2^g, V_3^g and V_4^g we arbitrarily discard some extra vertices to restore the balance in the 4-partite graphs.

Similarly we will show that either we can increase the size of our cover or we have

$$d_4\left(V_2^g \cup V_3^g \cup V_4^g, \binom{\mathcal{I}_t}{3}\right) \leq \sqrt{\gamma}. \tag{7}$$

Indeed assume the contrary, i.e. $d_4(V_2^g \cup V_3^g \cup V_4^g, \binom{\mathcal{I}_t}{3}) \geq \sqrt{\gamma}$, then since both $|\mathcal{I}_t|$ and $|V_2 \cup V_3^g \cup V_4^g|$ are at least γn, by Lemma 4 we can find disjoint complete 4-partite graphs with one color class in $V_2 \cup V_3^g \cup V_4^g$ and three in $\mathcal{I}_t$ covering at least $\gamma^2 n/2$ vertices. And again as above we can add these 4-partite graphs to our cover and increase the size of our cover as we have extra vertices in $V_2^g \cup V_3^g \cup V_4^g$. By the same reasoning we can prove that there are very few edges that uses two vertices from $V_2 \cup V_3^g \cup V_4^g$ and two from $\mathcal{I}_t$.

From the above observations about the size of $\mathcal{F}_g$ and (2) we have that $|V_2^g \cup V_3^g \cup V_4^g \cup \mathcal{I}_t| \geq (3/4 - \alpha)n$. Therefore if we can not increase the size of our cover significantly (by at least $\gamma^2 n/16$ vertices), then by (4), (6) and (7) we get that $d_4(V_2^g \cup V_3^g \cup V_4^g \cup \mathcal{I}_t) < \alpha$. Hence H is α-extremal.

5 The Extremal Case

Here our graph H is in the Extremal Case, i.e. *there is a $B \subset V(H)$ such that*

- $|B| \geq (\frac{3}{4} - \alpha)n$

- $d_4(B) < \alpha$.

We assume that n is sufficiently large and α is a sufficiently small constant < 1. Let $A = V(H) \setminus B$, by shifting some vertices between A and B we can have that $A = n/4$ and $B = 3n/4$ as $n \in 4\mathbb{Z}$ (we still keep the notation A and B). It is easy to see that we still have

$$d_4(B) < 6\alpha \tag{8}$$

Since we have

$$\delta_1(H) \geq \binom{n-1}{3} - \binom{3n/4}{3} + 1 = \binom{n-1}{3} - \binom{|B|}{3} + 1$$

this together with (8) implies that almost all 4-sets of $V(H)$ are edges of H except 4-sets of B. Thus roughly speaking we have that almost every vertex $b \in B$ makes edges with almost all 3-sets of vertices in $\binom{A}{3}$, with almost all 3-sets of vertices in $B \setminus \{b\} \times \binom{A}{2}$ and with almost all 3-sets of vertices in $\binom{B \setminus \{b\}}{2} \times A$ and vice versa. Therefore, we will basically match every vertex in A with a distinct 3-set of vertices in $\binom{B}{3}$ (disjoint from all 3-sets matched with other vertices in A) to get the perfect matching. However some vertices may be '*atypical*', in the sense that they may not have this connectivity structure hence we will first find a small matching that covers all such '*atypical*' vertices. For the remaining '*typical*' vertices we will show that they satisfy the conditions of König-Hall theorem, hence we will match every remaining vertex in A with a distinct 3-sets of remaining vertices in B.

A vertex $a \in A$ is called *exceptional* if it does not make edges with almost all 3-sets of vertices in B, more precisely if

$$deg_4\left(a, \binom{B}{3}\right) < (1 - \sqrt{\alpha})\binom{|B|}{3}$$

A vertex $a \in A$ is called *strongly exceptional* if it makes edges with very few 3-sets in B, more precisely if

$$deg_4\left(a, \binom{B}{3}\right) < \alpha^{1/3}\binom{|B|}{3}$$

28

Similarly a vertex $b \in B$ is called *exceptional* if it makes edges with many 3-sets of vertices in B more precisely if

$$deg_4\left(b, \binom{B \setminus \{b\}}{3}\right) > \sqrt{\alpha}\binom{|B|}{3}$$

A vertex $b \in B$ is called *strongly exceptional* if it makes edges with almost all 3-sets of vertices in B more precisely if

$$deg_4\left(b, \binom{B \setminus \{b\}}{3}\right) > (1 - \alpha^{1/3})\binom{|B|}{3}$$

Denote the set of *exceptional* and *strongly exceptional* vertices in A (and B) by X_A and SX_A respectively (similarly X_B and SX_B). Easy calculations using (1) and (8) yields that $|X_A| \leq 18\sqrt{\alpha}|A|$ and $|X_B| \leq 18\sqrt{\alpha}|B|$ and for the *strongly exceptional* sets we have $|SX_A| \leq 40\alpha|A|$ and $|SX_B| \leq 40\alpha|B|$. The constants are not the best possible but we choose them for ease of calculation.

If we have both SX_B and SX_A non empty, (say $b \in SX_B$ and $a \in SX_A$) then since then we can exchange a with b and reduce the size of both SX_B and SX_A, as it is easy to see that both a and b are not *strongly exceptional* in their new sets. Hence one of the sets SX_A and SX_B must be empty.

Assume $SX_B \neq \emptyset$. By definition of SX_B, for every vertex $b \in SX_B$, we have $deg_3(b, \binom{B}{3}) \geq (1 - \alpha^{1/3})\binom{|B|}{3}$. This together with the bound on the size of SX_B implies that we can greedily find $|SX_B|$ vertex disjoint edges in $H|_B$ each containing exactly one vertex of SX_B. We also select $|SX_B|$ other vertex disjoint edges such that each edge has two vertices in $B \setminus X_B$ and the two other vertices are in A. We can clearly find such edges because by (1) and definition of X_B every vertex in $B \setminus X_B$ makes edges with at least $(1 - 3\sqrt{\alpha})$-fraction of 3-sets in $B \times \binom{|A|}{2}$. We remove the vertices of these edges from A and B and denote the remaining set by A' and B'. Let $|A'| + |B'| = n'$, by the above procedure we have $n' = n - 8|SX_B|$, $|A'| = |A| - 2|SX_B|$ and $|B'| = |B| - 6|SX_B|$ hence we get $|B'| = 3|A'| = 3n'/4$.

In case $SX_A \neq \emptyset$ (and $SX_B = \emptyset$), we will first eliminate the vertices in SX_A. Note that in this case any vertex $b \in B$ is exchangeable with any vertex in SX_A, because if there is a vertex $b \in B$ such that $deg_4(b, \binom{B}{3}) \geq \alpha^{1/3}\binom{|B|}{3}$ then we can replace b with any vertex $a \in SX_A$ to reduce the size of SX_A (as the vertex b is not *strongly exceptional* in A and a can not be *strongly exceptional* in the set B). Therefore we consider the whole set $SX_A \cup B$. By (1) for any vertex $v \in SX_A \cup B$

we have

$$deg_4\left(v, \binom{SX_A \cup B}{3}\right) \geq (|SX_A| - 1)\binom{|B|}{2} + \binom{(|SX_A| - 1)}{2}|B| + \binom{(|SX_A| - 1)}{3} + 1$$

$$\geq \binom{4(|SX_A| - 1)}{3} + 1$$

where the last inequality holds when n is large enough and $|SX_A$ is small. So with a simple greedy procedure we find $|SX_A|$ disjoint edges in $H|_{SX_A \cup B}$ and remove these edges from H. Note that this is the only place where we critically use the minimum degree. We let $A' = A \setminus SX_A$ and B' has all other remaining vertices. Again as above we have $n' = n - 4|SX_A|$, $|A'| = |A| - |SX_A|$ and $|B'| = |B| - 3|SX_A|$ hence we get $|B'| = 2|A'| = 3n'/4$.

Having dealt with the *strongly exceptional* vertices, the vertices of X_A and X_B in A' and B' can be eliminated using the fact that their sizes are much smaller than the crossing degrees of vertices in those sets. For instance as observed above we have $|X_A| \leq 18\sqrt{\alpha}|A|$ while for any vertex $a \in X_A$, we have that $deg_4(a, \binom{B'}{2}) \geq \alpha^{1/3}\binom{|B'|}{3}/2$ (because $a \notin SX_A$). Therefore by a simple greedy procedure for each $a \in X_A$ we delete a disjoint edge that contains a and three vertices from B'. Similarly for each $b \in X_B$ we delete an edge that contains b and uses one vertex from A' and the other two vertices from B' distinct from b. Clearly we can find such disjoint edges, hence we removed a partial matching that covers all vertices in the *strongly exceptional* and *exceptional* sets.

Finally in the leftover sets of A' and B' (denote them by A'' and B'', by construction we still have $|B''| = 3|A''|$) we will find $|A''|$ disjoint edges each using one vertex in A'' and three vertices in B''. Note that for every vertex $a \in A''$ we have $deg_4(a, \binom{B''}{3}) \geq (1 - 2\alpha^{1/3})\binom{|B''|}{3}$ (as $a \notin X_A$). We say that a vertex b_i and a pair b_j, b_k in B'' are *good* for each other if $(b_i, b_j, b_k, a_l) \in E(H)$ for at least $(1 - 40\alpha^{1/4})|A''|$ vertices a_l in A''. We have that any vertex $b_i \in B''$ is *good* for at least $(1 - 40\alpha^{1/4})\binom{|B''|}{2}$ pairs of vertices in B'' (again this is so because $b_i \notin X_B$). We call such a (b_i, b_j, b_j) a good triplet.

We randomly select a set T_1 of $100\alpha^{1/4}|B''|$ vertex disjoint 3-sets of vertices in B''. By the above observation with high probability every vertex $a \in A''$ make edges in H with at least $3|T_1|/4$ triplets in T_1 and every triplet in T_1 makes an edge with at least $3|A''|/4$ vertices in A''. In $B'' \setminus V(T_1)$ still every vertex is *good* for almost all pairs (as the size of T_1 is very small).

We cover vertices in $B'' \setminus V(T_1)$ with disjoint good triplets (i.e. the triplet makes an edge in H with at least $(1 - 40\alpha^{1/4})|A''|$ vertices in A''. This can be done by considering a 3-graph with vertex set $B'' \setminus V(T_1)$ and all the *good* triplets as its edges. As argued above every vertex is good for almost all pairs. We can find a perfect matching in this 3-graph (see [13]). Let the set of triplets in this perfect matching be T_2.

Now construct an auxiliary bipartite graph $G(L, R)$, such that $L = A''$ and vertices in R corresponds to the triplets in T_1 and T_2. A vertex in $a_l \in L$ is connected to a vertex $y \in R$ if the triplet corresponding to y (say b_i, b_j, b_k) is such that $(b_i, b_j, b_k, a_l) \in E(H)$. We will show that $G(L, R)$ satisfies the König-Hall criteria. Considering the sizes of A'' and T_1 it is easy to see that for every subset $Q \subset R$ if $|Q| \leq (1 - 40\alpha^{1/4})|A''|$ then $|N(Q)| \geq |Q|$. When $|Q| > (1 - 40\alpha^{1/4})|A''|$ (using $|B''| = 2|A''|$) any such Q must have at least $6|T_1|/10$ vertices corresponding to pairs in T_1, hence with high probability $N(Q) = L \geq |Q|$. Therefore there is a perfect matching of R into L. This perfect matching in $G(L, R)$ readily gives us a matching in H covering all vertices in A'' and B'', which together with the edges we already removed (covering *strongly exceptional* and *exceptional* vertices) is a perfect matching in H. □

References

[1] B. Bollobás, *Extremal Graph Theory*, Academic Press, London (1978).

[2] H. Hàn, Y. Person and M. Schacht, On Perfect Matchings in Uniform Hypergraphs with Large Minimum Vertex Degree, *SIAM J. Discret. Math.*, 23(2), (2009), pp. 732–748.

[3] P. Kővári, V.T. Sós, P. Turán, On a problem of Zarankiewicz, *Colloq. Math.*, 3 (1954), pp. 50-57.

[4] P. Erdös, On extremal problems of graphs and generalized graphs, *Israel J. Math.* 2, (1964), 183–190.

[5] D. Kühn and D. Osthus, Matchings in hypergraphs of large minimum degree, *J. Graph Theory*, 51(4),(2006), pp. 269–280

[6] D. Kühn, D. Osthus and A. Treglown, Matchings in 3-uniform hypergraphs, (submitted).

[7] O. Pikhurko, Perfect matchings and K3 4-tilings in hypergraphs of large codegree, *Graphs Combin.* 24(4) (2008) 391-404.

[8] V. Rödl, A. Ruciński and E. Szemerédi, A Dirac-Type Theorem for 3-Uniform Hypergraphs, *Comb. Probab. Comput.*, Volume 15, (2006), pp. 229–251.

[9] V. Rödl, A. Ruciński and E. Szemerédi, Perfect matchings in uniform hypergraphs with large minimum degree, *Europ. J. Combin.* 27 (2006) 1333-1349.

[10] V. Rödl, A. Ruciński and E. Szemerédi, An approximate Dirac-type theorem for k-uniform hypergraphs, *Combinatorica* 28(2) (2008) 229-260.

[11] V. Rödl, A. Ruciński and E. Szemerédi, A note on perfect matchings in uniform hypergraphs with large minimum collective degree, *Commentationes Mathematicae Universitatis Carolinae* 49(4) (2008) 633-636.

[12] V. Rödl, A. Ruciński and E. Szemerédi, Perfect matchings in large uniform hypergraphs with large minimum collective degree, *J. Comb. Theory Ser. A*, 116(3), (2009), pp. 613–636.

[13] I. Khan, Perfect matching in 3 uniform hypergraphs with large vertex degree, *submitted*

[14] A. Ruciński and K. Markström Perfect matchings and Hamilton cycles in hypergraphs with large degrees *Submitted*

[15] V. Rödl and A. Ruciński, Dirac-type questions for hypergraphs – a survey (or more problems for Endre to solve), An Irregular Mind (Szemerédi is 70), Bolyai Soc. Math. Studies 21 (2010)