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Perfect matchings in large uniform hypergraphs with large minimum collective degree

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ABSTRACT

We define a perfect matching in a k-uniform hypergraph H on n vertices as a set of $\lfloor n/k \rfloor$ disjoint edges. Let $\delta_{k-1}(H)$ be the largest integer d such that every $(k-1)$-element set of vertices of H belongs to at least d edges of H.

In this paper we study the relation between $\delta_{k-1}(H)$ and the presence of a perfect matching in H for $k \geqslant 3$. Let $t(k,n)$ be the smallest integer t such that every k-uniform hypergraph on n vertices and with $\delta_{k-1}(H) \geqslant t$ contains a perfect matching.

For large n divisible by k, we completely determine the values of $t(k,n)$, which turn out to be very close to $n/2 - k$. For example, if k is odd and n is large and even, then $t(k,n) = n/2 - k + 2$. In contrast, for n not divisible by k, we show that $t(k,n) \sim n/k$.

In the proofs we employ a newly developed "absorbing" technique, which has a potential to be applicable in a more general context of establishing existence of spanning subgraphs of graphs and hypergraphs.

1. Introduction

A k-uniform hypergraph is a pair $H = (V, E)$, where $V := V(H)$ is a finite set of vertices and $E := E(H) \subseteq \binom{V}{k}$ is a family of k-element subsets of V. Whenever convenient we will identify H with $E(H)$.

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A matching in H is a set of disjoint edges of H. A matching of size ℓ will be sometimes called *an ℓ-matching*. For $V' \subseteq V$, $H[V']$ is the sub-hypergraph of H with $V(H') = V'$ and $E(H') = \{e \in E\colon\ e \subseteq V'\}$. We will also use the shorthand notation $H - V' := H[V \setminus V']$ and $H - v := H - \{v\}$.

Given a k-uniform hypergraph H and r vertices $v_1, \dots, v_r \in V(H)$, $1 \leqslant r \leqslant k-1$, we denote by $\deg_H(v_1, \dots, v_r)$ the number of edges of H which contain $v_1, \dots, v_r$. Let $\delta_r(H) := \delta_r$ be the minimum of $\deg_H(v_1, \dots, v_r)$ over all r-element sets of vertices of H.

Definition 1.1. For all integers $k \geqslant 2$ and $n \geqslant k$, denote by $t(k,n)$ the smallest integer t such that every k-uniform hypergraph on n vertices and with $\delta_{k-1} \geqslant t$ contains a matching of size $\lfloor n/k \rfloor$.

Assume first that n is divisible by k (*the divisible case*). For graphs, an easy argument shows that $t(2,n) = n/2$. The goal of this paper is to determine $t(k,n)$ for $k \geqslant 3$. It follows from [7] that $t(k,n) \leqslant n/2 + o(n)$. In [4], Kühn and Osthus proved that $t(k,n) \leqslant n/2 + 3k^2\sqrt{n \log n}$. This was further improved in [5] to $t(k,n) \leqslant n/2 + C \log n$. Recently, together with Mathias Schacht we observed that an argument from [1] can be modified to yield $t(k,n) \leqslant n/2 + k/4$.

As for the lower bound on $t(k,n)$, Kühn and Osthus gave constructions of hypergraphs $H^0(k,n)$, which yield that $t(k,n) \geqslant n/2 + 1 - k$ when both k and $n/2$ are even, and $t(k,n) \geqslant n/2 + 2 - k$ when both k and $n/2$ are odd (see [4, Lemma 15]).

Here we extend their constructions to all other cases and slightly improve their lower bounds when k and $n/2$ are both even. Our main result, however, is the upper bound which shows that these constructions are optimal.

Theorem 1.1. *For all $k \geqslant 3$ and sufficiently large n divisible by k,*

$$t(k,n) = \begin{cases} n/2 + 3 - k & \text{if } k/2 \text{ is even and } n/k \text{ is odd,} \\ n/2 + 5/2 - k & \text{if } k \text{ is odd and } (n-1)/2 \text{ is odd,} \\ n/2 + 3/2 - k & \text{if } k \text{ is odd and } (n-1)/2 \text{ is even,} \\ n/2 + 2 - k & \text{otherwise}. \end{cases} \tag{1}$$

In a less compact way, the last case above ($t(k,n) = n/2 + 2 - k$) holds when

- k is even and n/k is even,
- $k/2$ is odd and n/k is odd,
- k is odd and n is even.

In Section 3, we describe critical constructions which yield the lower bounds in (1). The proof of the upper bounds is presented in Sections 4 and 5.

Let us remark that our Theorem 1.1 shows that $n/2 - t(k,n)$ is roughly equal to k. It is perhaps interesting to note that for a similar problem regarding Hamiltonian cycles in k-uniform hypergraphs, the corresponding difference is at most $k/2$ (see [3,7]).

For n not divisible by k (*the non-divisible case*), the values of $t(k,n)$ are substantially smaller than in the divisible case (see Proposition 2.1 in Section 2).

Our proofs of upper bounds on $t(k,n)$ involve a recently developed (see [6,5,7]) "absorbing" technique. For the divisible case, roughly speaking, in order to construct a perfect matching, we begin with a powerful, but relatively small matching M'. This matching has the property that for any set S of k unmatched vertices, one can slightly alter M', so that the resulting matching M'_S contains precisely the vertices of the set $V(M') \cup S$, i.e., $V(M'_S) = V(M') \cup S$. With M' constructed, we next find an *almost* perfect matching M'' in $H - V(M')$ which leaves some set S of k vertices unmatched. The matching $M'_S \cup M''$ is then perfect. In the non-divisible case, this technique is suitably modified (see Section 2).

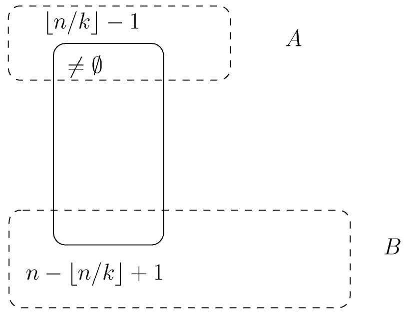

Fig. 1. The construction yielding the lower bound in Proposition 2.1.

2. The non-divisible case

In this section we study the threshold $t(k,n)$ in the non-divisible case. Interestingly, in this setting the threshold drops dramatically from about $n/2$ in the divisible case (cf. Theorem 1.1) down to about n/k.

Unlike the divisible case, here we just give a fairly simple proof of an upper bound on $t(k,n)$. Although it does not match the lower bound, somewhat surprisingly, it is later used in the proof of Theorem 1.1, which does yield a precise value of $t(k,n)$ when k divides n.

Proposition 2.1. *If $n \not\equiv 0 \pmod k$, then*

$$\left\lfloor \frac{n}{k} \right\rfloor \leqslant t(k,n) \leqslant \frac{n}{k} + O(\log n).$$

The lower bound follows immediately from considering the k-uniform hypergraph whose vertex set is split into two sets, A and B, where $|A| = \lfloor n/k \rfloor - 1$, and the edge set consists of all k-element sets which contain at least one vertex of A (see Fig. 1).

For the proof of the upper bound in Proposition 2.1, we need a couple of facts. A variant of the first one has been proved in [5, Proposition 2.1]. Let $\nu(H)$ be the size of a maximum matching in H.

Fact 2.1. *Let $n \geqslant k \geqslant 2$. For every k-uniform hypergraph H on n vertices,*

$$\nu(H) \geqslant \min\left\{ \left\lfloor \frac{n}{k} \right\rfloor - k + 2, \delta_{k-1}(H) \right\}.$$

In particular, if $\delta_{k-1}(H) \geqslant \lfloor n/k \rfloor - k + 2$, then there is a matching in H covering all but at most $(k-2)k + r$ vertices, where $r \equiv n \pmod k$.

Proof. Let $d = \min\{\lfloor n/k \rfloor - k + 2, \delta_{k-1}(H)\}$. If $d = 0$, there is nothing to prove. If $d = 1$, then $\delta_{k-1}(H) \geqslant 1$, hence $\nu(H) \geqslant 1$. Assume that $d \geqslant 2$ and suppose that a largest matching M has size $|M| \leqslant d - 1$. Then there are at least $(k-1)k$ vertices unmatched by M.

Let us select arbitrarily k disjoint sets $S_1, \ldots, S_k$ of size $|S_i| = k - 1$ from among the vertices unmatched by M. Each of these sets has at least $\delta_{k-1}(H)$ neighbors v and all these neighbors belong to $V(M)$. Thus, there are at least $k\delta_{k-1}(H)$ pairs of the form (S_i, v), where $v \in V(M)$ and $S_i \cup \{v\} \in H$.

By averaging, there is an edge $e \in M$ such that at least $k\delta_{k-1}(H)/(d-1)$, that is, at least $k+1$ of these pairs have $v \in e$. This implies, however, that there are indices $1 \leqslant i < j \leqslant k$ and distinct vertices $v', v'' \in e$ such that $e' := S_i \cup \{v'\} \in H$ and $e'' := S_j \cup \{v''\} \in H$. Swapping e for e' and e'' brings us to a contradiction with the maximality of M (see Fig. 2). □

The main idea of the proof of the upper bound in Proposition 2.1, as well as of Theorem 1.1, is to use an “absorbing device.” Because in Proposition 2.1 n is not divisible by k, there are always at least

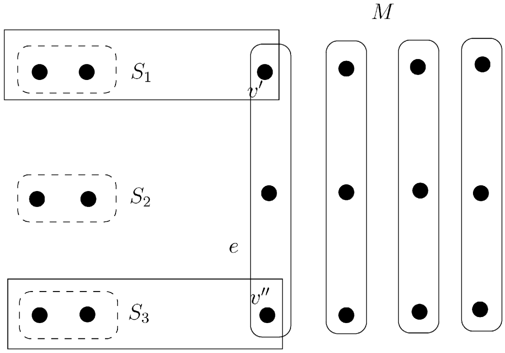

Fig. 2. Illustration to the proof of Fact 2.1 ($k = 3$).

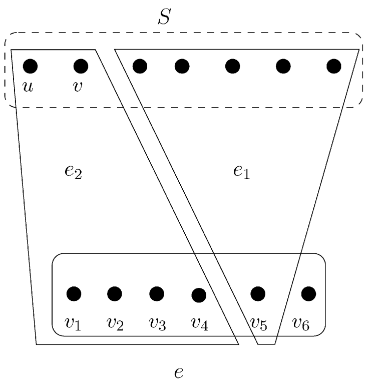

Fig. 3. An S-absorbing edge e ($k = 6$).

$k + 1$ vertices outside a matching of size less than $\lfloor n/k \rfloor$. Hence, in a single instance of absorption, we have the liberty to include $k + 1$ vertices into a matching (and exclude one). This idea is facilitated as follows.

Definition 2.1. Given a set S of $k + 1$ vertices, we call an edge $e \in H$ disjoint from S S-*absorbing* if there are two disjoint edges e_1 and e_2 in H such that $|e_1 \cap S| = k - 1$, $|e_1 \cap e| = 1$, $|e_2 \cap S| = 2$ and $|e_2 \cap e| = k - 2$ (see Fig. 3). For fixed e, e_1, e_2, we denote the matching $M' \setminus \{e\} \cup \{e_1, e_2\}$ by M'_S.

The idea of S-absorbing edges will be exploited as follows. Let S be a set of $k + 1$ vertices and let M' be a matching, where $V(M') \cap S = \emptyset$, which contains an S-absorbing edge e. Then M' can "absorb" S by swapping e for e_1 and e_2 (one vertex of e will become unmatched), that is, by replacing M' by M'_S.

Fact 2.2. *If for some $c > 0$ we have $\delta_{k-1}(H) \geqslant cn$ then for large n and for every $(k+1)$-element set of vertices S, the number of S-absorbing edges e is at least $\frac{1}{2}c^3 n^k/k!$.*

Proof. Let us fix two vertices u and v in S and count only those S-absorbing edges e for which the corresponding edge e_2 contains u and v. To prove the estimate in Fact 2.2, we will count ordered k-tuples of *distinct* vertices $(v_1, \ldots, v_k)$ such that $e = \{v_1, \ldots, v_k\}$ is disjoint from S, $e_1 \cap e = \{v_{k-1}\}$, and $e_2 = \{v_1, \ldots, v_{k-2}, u, v\}$, and divide the result by $k!$.

For each $j = 1, \ldots, k - 3$, there are precisely $n - j - k$ choices of vertex v_j. Having selected $v_1, \ldots, v_{k-3}$, each of v_{k-2}, v_{k-1} and v_k must be a neighbor of an already fixed $(k - 1)$-tuple of vertices (see Fig. 3). Thus, there are at least $\delta_{k-1}(H) - j - 1$ choices of v_j, $j = k-2, k-1, k$. Altogether, there are at least $(n-2k)^{k-3}$ choices of $v_1, \dots, v_{k-3}$, followed by at least $(\delta_{k-1}(H) - 2k)^3$ choices of v_{k-2}, v_{k-1}, v_k. For large n, this yields the required bound. □

Fact 2.3. *For all $c > 0$ there exist $C > 0$ and n_0 such that if H is a k-uniform hypergraph with $n \geqslant n_0$ vertices and $\delta_{k-1}(H) \geqslant cn$, then there exists a matching M' in H of size $|M'| \leqslant C \log n$ and such that for every $(k+1)$-tuple S of vertices of H, the number of S-absorbing edges in M' is at least $k-2$.*

Proof. The proof is probabilistic. Select a random subset M' of H, where each edge is chosen independently with probability $p = C(\log n)n^{-k}$. Then, the expected size of M' is at most $\binom{n}{k}p < n^k p/k!$, and the expected number of intersecting pairs of edges in M' is at most $n^{2k-1}p^2 = o(1)$. Hence, by Markov's inequality, M' is a matching of size at most $C \log n$ with probability at least $1 - 1/k! - o(1)$.

For every $(k+1)$-tuple of vertices S, let X_S be the number of S-absorbing edges in M'. Then, by Fact 2.2,

$$\mathbb{E}X_S \geqslant \frac{c^3 n^k p}{2k!} = \frac{Cc^3 \log n}{2k!}.$$

By Chernoff's bound (see, e.g., [2, Theorem 2.1]),

$$\mathbb{P}\left(X_S \leqslant \frac{1}{2}\mathbb{E}X_S\right) \leqslant \exp\left\{-\frac{1}{8}\mathbb{E}X_S\right\} \leqslant \exp\left\{-\frac{Cc^3 \log n}{16k!}\right\} = o\left(n^{-k-1}\right)$$

if $C > 16(k+1)!/c^3$. Thus, with probability $1 - o(1)$, for each $(k+1)$-tuple S of the vertices in $V(H)$, there are at least

$$\frac{1}{4k!}c^3 C \log n > 4(k+1)\log n \gg k-2$$

S-absorbing edges in M'. In conclusion, with positive probability M' has all the properties listed in the statement of Fact 2.3, and thus such a matching M' exists. □

Now we are ready to give a short proof of the upper bound in Proposition 2.1.

Proof of Proposition 2.1. Let H be a k-uniform hypergraph with $\delta_{k-1}(H) \geqslant n/k + Ck \log n$ and n vertices, $n \geqslant n_0$, where C and n_0 are determined by Fact 2.3 with $c = 1/k$. Let M' be a matching in H as described in Fact 2.3 and let $H' = H - V(M')$ be the sub-hypergraph of H induced by $V \setminus V(M')$. Note that $n' := |V(H')| = n - k|M'|$, $|M'| \leqslant C \log n$, and

$$\delta_{k-1}(H') \geqslant \delta_{k-1}(H) - Ck \log n \geqslant \frac{n}{k} > \frac{n'}{k}.$$

Thus, by Fact 2.1, there is in H' a matching M'' of size $|M''| \geqslant \lfloor n'/k \rfloor - k + 2$. This matching leaves at most $(k-2)k + k - 1$ vertices of H' uncovered.

Let $T = V(H) \setminus V(M' \cup M'')$, $|T| \leqslant (k-2)k + k - 1$. As long as the number of uncovered vertices remains at least $k+1$, we repeat the following absorbing procedure. Recall that for every $(k+1)$-tuple S of the vertices of H, the number of S-absorbing edges in M' is at least $k-2$. Take a set S of $k+1$ uncovered vertices and find in M' an S-absorbing edge e. Replace M' by $M'_S := M' \setminus \{e\} \cup \{e_1, e_2\}$ (see Definition 2.1), decreasing the number of uncovered vertices by k. Since we only have at most $k-2$ iterations, there will always be an available (unused) S-absorbing edge in M'. In the end, we obtain a matching $\overline{M'}$ with $V(\overline{M'}) \subseteq V(M') \cup T$ and $|\overline{M'}| = |M'| + \lfloor |T|/k \rfloor$. But then $|\overline{M'} \cup M''| = \lfloor n/k \rfloor$ as needed. □

We conjecture that for all $k \geqslant 3$, if $n \not\equiv 0 \pmod{k}$ then $t(k,n) = \lfloor n/k \rfloor$.

Remark 2.1. For $\ell \geqslant 1$ and $n \geqslant k\ell$, let $t^{(\ell)}(k,n)$ be the smallest integer t such that for every k-uniform hypergraph H on n vertices and with $\delta_{k-1}(H) \geqslant t$, we have $\nu(H) \geqslant \lfloor n/k \rfloor - \ell$. Using Fact 2.1 and

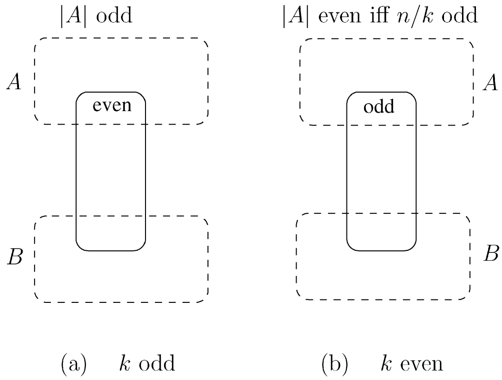

Fig. 4. Critical hypergraphs $H^0(k,n)$.

Proposition 2.1, it is easy to prove that $t^{(\ell)}(k,n) = \lfloor n/k \rfloor - \ell$ for $\ell \geqslant k-2$, while for $\ell \leqslant k-3$ we have $\lfloor n/k \rfloor - \ell \leqslant t^{(\ell)}(k,n) \leqslant t(k,n) \leqslant n/k + O(\log n)$.

3. Critical constructions and the outline of the main proof

From now on, we assume that n is divisible by k. In this section we begin by describing a family of *critical* hypergraphs, $H^0(k,n)$, which establish the lower bounds in Theorem 1.1 and play a crucial role in the proof of the upper bounds too. Then, we state two lemmas and show how they together imply Theorem 1.1.

3.1. Critical constructions

We define critical hypergraphs separately for odd and even k and notice that in the odd case this is the construction from [4].

Definition 3.1 *(odd k).* Let k be odd. *The n-vertex set V* of $H^0(k,n)$ is divided into two subsets, A and B, where $|A| := a(k,n)$ is the unique *odd* integer from the set

$$\left\{\frac{n}{2}-1, \frac{n}{2}-\frac{1}{2}, \frac{n}{2}, \frac{n}{2}+\frac{1}{2}\right\}.$$

The edge set of $H^0(k,n)$ consists of all k-element sets intersecting A in an *even* number of vertices (see Fig. 4(a)).

Since $|A|$ is odd while each edge intersects A in an even number of vertices, no matching of $H^0(k,n)$ can cover A and thus, $H^0(k,n)$ has no perfect matching. Moreover, it is easy to check that for odd k, $\delta_{k-1}(H^0(k,n)) = \min(|A|-k+2, |B|-k+1)$, hence

$$\delta^0 := \delta_{k-1}\big(H^0(k,n)\big) = \begin{cases} \frac{n}{2}+1-k & \text{if } n=4m, \\ \frac{n}{2}+\frac{1}{2}-k & \text{if } n=4m+1, \\ \frac{n}{2}+1-k & \text{if } n=4m+2, \\ \frac{n}{2}+\frac{3}{2}-k & \text{if } n=4m+3. \end{cases} \tag{2}$$

For even k, define the hypergraph $H^0(k,n)$ as follows.

Definition 3.2 *(even k).* Let k be even. *The n-vertex set V* of $H^0(k,n)$ is divided into two subsets, A and B, where

$$|A| := a(k,n) = \begin{cases} \frac{n}{2} - 1 & \text{if } \frac{n}{k} \text{ is even,} \\ \frac{n}{2} - 1 & \text{if } \frac{n}{k} \text{ is odd and } \frac{n}{2} \text{ is odd,} \\ \frac{n}{2} & \text{if } \frac{n}{k} \text{ is odd and } \frac{n}{2} \text{ is even.} \end{cases}$$

The edge set of $H^0(k,n)$ consists of all k-element sets intersecting A in an *odd* number of vertices (see Fig. 4(b)).

For even k, $H^0(k,n)$ has no perfect matching either. Indeed, if n/k is even then $n/2$ is even, and so $|A|$ is odd. Consequently, A cannot be perfectly covered by an even number of odd sets. On the other hand, if n/k is odd then $|A|$ is even, and thus it is impossible to cover A by an odd number of odd sets. Moreover, it is easy to see that when k is even, $\delta_{k-1}(H^0(k,n)) = |A| - k + 2$, hence the three cases defining the size of A in Definition 3.2 yield three corresponding cases for the value of $\delta_{k-1}(H^0(k,n))$:

$$\delta^0 := \delta_{k-1}\big(H^0(k,n)\big) = \begin{cases} \frac{n}{2} + 1 - k & \text{if } n = 2mk, \\ \frac{n}{2} + 1 - k & \text{if } n = (2m+1)k \text{ and } k = 4l + 2, \\ \frac{n}{2} + 2 - k & \text{if } n = (2m+1)k \text{ and } k = 4l. \end{cases} \tag{3}$$

3.2. *The outline of the main proof*

For all k and n set

$$\delta^0(k,n) = \delta_{k-1}\big(H^0(k,n)\big).$$

A careful case by case comparison verifies that $\delta^0(k,n) + 1$ is equal to the quantity appearing in (1), and thus our Theorem 1.1 can be restated as follows.

Theorem 3.1. *For all $k \geqslant 3$ and sufficiently large n divisible by k,*

$$t(k,n) = \delta^0(k,n) + 1.$$

Since $H^0(k,n)$ does not have a perfect matching, $t(k,n) \geqslant \delta^0(k,n) + 1$, and it remains to prove the opposite inequality. In other words, in order to show Theorem 3.1, and thus Theorem 1.1, it is enough to prove that if

$$\delta_{k-1}(H) \geqslant \delta^0(k,n) + 1, \tag{4}$$

and if n is large and divisible by k, then H has a perfect matching.

We will consider two cases separately: when H is almost completely contains the critical hypergraph $H^0(k,n)$ or its complement $\overline{H^0(k,n)}$ (Section 4), and when it does not (Section 5). In the former case we will find a perfect matching "manually," relying heavily on the structure of the critical hypergraphs. In the latter case, we will employ the absorbing technique similar to that used already in the non-divisible case (see Section 2).

Definition 3.3. Given two k-uniform n-vertex hypergraphs, H and H^0, we denote by $c(H, H^0)$ the minimum of $|E(H^0) \setminus E(H')|$ taken over all isomorphic copies H' of H with the vertex set $V(H') = V(H^0)$. We say that H *ε-contains* H_0 and write $H_0 \subset_\varepsilon H$ if $c(H, H^0) < \varepsilon n^k$.

Note that $c(H, H^0)$ is typically different from $c(H^0, H)$. A small value of $c(H, H^0)$ means that one can find a large part of H^0 inside H.

As the first major step toward proving Theorem 3.1, we will show in Section 4 the following lemma.

Lemma 3.1. *There exist $\epsilon > 0$ and n_0 such that if*

- *H is a k-uniform hypergraph on $n > n_0$ vertices, n divisible by $k \geqslant 3$,*
- *$\delta_{k-1}(H) \geqslant \delta^0(k,n) + 1$, and*
- *$H^0(k,n) \subset_\varepsilon H$ or $\overline{H^0(k,n)} \subset_\varepsilon H$,*

then H has a perfect matching.

This lemma will be then complemented in Section 5 by the following result.

Lemma 3.2. *For all $\epsilon > 0$ there exists n_0 such that if*

- *H is a k-uniform hypergraph on $n > n_0$ vertices, n divisible by $k \geqslant 3$,*
- *$\delta_{k-1}(H) \geqslant (1/2 - 1/\log n)n$, and*
- *$H^0(k,n) \not\subset_\varepsilon H$ and $\overline{H^0(k,n)} \not\subset_\varepsilon H$,*

then H has a perfect matching.

Note that in Lemma 3.2, the existence of a perfect matching is guaranteed by a weaker degree assumption than in Lemma 3.1. In fact, with an extra computational effort, we could replace the term $1/\log n$ in the second assumption of Lemma 3.2 by a small constant $\gamma > 0$.

Taking the above two lemmas for granted, the proof of Theorem 3.1 is immediate.

Proof of Theorem 3.1. Let $\epsilon > 0$ be so small that Lemma 3.1 holds, and let n be so large that both Lemmas 3.1 and 3.2 hold with this ε. Let H be a k-uniform hypergraph on $n > n_0$ vertices, n divisible by k, which satisfies the degree condition (4). If $H^0(k,n) \subset_\varepsilon H$ or $\overline{H^0(k,n)} \subset_\varepsilon H$, apply Lemma 3.1, and otherwise apply Lemma 3.2. In either case, H contains a perfect matching. □

4. Near the critical construction

In this section we prove Lemma 3.1, that is, we show that hypergraphs H which ε-contain the critical hypergraph $H^0(k,n)$ or its complement, but satisfy the degree condition (4), contain a perfect matching.

4.1. Preliminaries

In the proof of Lemma 3.1 we will use a couple of facts about a special kind of k-uniform hypergraphs. Given a bipartition $W = C \cup D$ and integers $k \geqslant 3$ and $0 \leqslant r \leqslant k$, a k-uniform hypergraph F with vertex set $V(F) = W$ is called *$(r, k-r)$-bipartite* if for every edge $e \in F$, we have $|e \cap C| = r$ (and thus $|e \cap D| = k - r$). If F consists of all k-element subsets of W which contain precisely r vertices of C, then we call F *complete $(r, k-r)$-bipartite* and denote it by $K_r(C,D) := K_r$. For future references, notice that

$$\deg_{K_r}(v) = \begin{cases} \binom{|C|}{r-1}\binom{|D|}{k-r} & \text{if } v \in C, \\ \binom{|C|}{r}\binom{|D|}{k-r-1} & \text{if } v \in D. \end{cases} \tag{5}$$

In Facts 4.1–4.3 below, we implicitly assume that

- $k \geqslant 3$ and $0 \leqslant r \leqslant k$,
- F is $(r, k-r)$-bipartite with $W = C \cup D$, $|W| = m$ and $|C|, |D| \geqslant 0.4m$,
- $\varepsilon = \varepsilon(k) > 0$ is sufficiently small.

The first fact establishes the existence of a perfect matching in F if all vertex degrees are very high.

Definition 4.1. A vertex $v \in W$ is called *ε-deficient* in F if

$$\deg_F(v) \leqslant \deg_{K_r}(v) - \varepsilon m^{k-1}.$$

Note that for $r = 0$, no vertex of $v \in C$ is ε-deficient for any $\varepsilon > 0$, and the same is true for $r = k$ and $v \in D$. The next observation follows directly from the definition.

Observation 1. *Let $c > 0$. If v is not ε-deficient in F and $W' = C' \cup D'$, where $|C'| \geqslant c|C|$ and $|D'| \geqslant c|D|$, then v is not ε/c^{k-1}-deficient in the induced sub-hypergraph $F[W']$.*

Fact 4.1. *If for some integer t we have $m = tk$, $|C| = tr$, $|D| = t(k-r)$, and no vertex is ε-deficient in F, then F has a perfect matching.*

Proof. Let M be a largest matching in F. Since $W \setminus V(M)$ is an independent set in F, we must have $|M| \geqslant m/(2k)$, since otherwise, for sufficiently small ε, every vertex of $W \setminus V(M)$ would be ε-deficient in F.

Suppose that a set of k vertices $S = \{v_1, \dots, v_k\}$ is not covered by M. Call a family $\{e_1, \dots, e_{k-1}\}$ of $k-1$ edges of M *S-complete* if the set $\bigcup_{i=1}^{k-1} e_i \cup S$ induces in F a complete k-partite sub-hypergraph. More precisely, every set T such that $|T \cap e_i| = |T \cap S| = 1$ for $i = 1, \dots, k-1$, and $|T \cap C| = r$, is an edge of F.

There are at least $\binom{m/(2k)}{k-1}$ families $\{e_1, \dots, e_{k-1}\}$ available, out of which at most $k\varepsilon m^{k-1}$ may not be S-complete (this is the total maximum degree deficiency of the vertices in S). Hence, for sufficiently small ε, at least one S-complete family exists. But having an S-complete family, we may replace the $k-1$ edges $e_1, \dots, e_{k-1}$ of M by k new disjoint edges, covering the vertices in the set $e_1 \cup \dots \cup e_{k-1} \cup S$ – a contradiction with the maximality of M. $\square$

If F is almost complete then the number of deficient vertices in F must be small. We make this observation precise.

Fact 4.2. *If $|K_r \setminus F| < \varepsilon m^k$ then the number of $\sqrt{\varepsilon}$-deficient vertices in F is at most $\sqrt{\varepsilon}km$.*

Proof. Suppose that more than $\sqrt{\varepsilon}km$ vertices of W are $\sqrt{\varepsilon}$-deficient in F. Since for each such a vertex v we have $\deg_{K_r - F}(v) \geqslant \sqrt{\varepsilon}m^{k-1}$, and since each edge is counted at most k times,

$$|K_r \setminus F| > \sqrt{\varepsilon}km \times \sqrt{\varepsilon}m^{k-1} \times \frac{1}{k} = \varepsilon m^k,$$

a contradiction with our assumption. $\square$

We will also distinguish another category of small degree vertices.

Definition 4.2. For $0 < c < 1$, call a vertex v *c-small* in F if

$$\deg_F(v) \leqslant c \times \deg_{K_r}(v),$$

and call it *c-large* otherwise.

Note that for fixed c and sufficiently small ε every c-small vertex is also ε-deficient. Hence, we have the following immediate consequence of Fact 4.2.

Corollary 4.1. *For every $0 < c < 1$, if $|K_r \setminus F| < \varepsilon m^k$ then the number of c-small vertices in F is at most $\sqrt{\varepsilon}km$.*

It is easy to cover small sets of large vertices by a matching.

Fact 4.3. *Let N be a set of 0.1-large vertices in F. If $|N| \leqslant \varepsilon m$ then there is a matching $M(N)$ in F whose every edge contains precisely one vertex of N.*

Proof. Let M be a matching in F covering a largest number of vertices of N and such that for every $e \in M$, $|e \cap N| = 1$. Suppose $N \setminus V(M) \neq \emptyset$ and fix $v \in N \setminus V(M)$. Since $|N \cup V(M)| \leqslant \varepsilon km$, there are at most εkm^{k-1} edges $e \in F$ such that $v \in e$ and $(e \setminus \{v\}) \cap (N \cup V(M)) \neq \emptyset$. On the other hand, since v is 0.1-large and $|C|, |D| \geqslant 0.4m$, we have by (5)

$$\deg_F(v) > 0.1 \deg_{K_r}(v) > 0.1 \frac{(m/3)^{k-1}}{(r-1)!(k-r)!} > \varepsilon km^{k-1}.$$

Consequently, there is an edge $e \ni v$ such that $(e \setminus \{v\}) \cap (N \cup V(M)) = \emptyset$, and the matching M can be extended to cover v, a contradiction with the choice of M. □

Going back to the set-up of Lemma 3.1, it will be convenient to view the critical hypergraphs as unions of complete bipartite hypergraphs $K_r(A, B) := K_r$, where $|A| = a(k, n)$ and $|B| = n - |A|$, are the sizes of the partition classes of the critical constructions defined in Definitions 3.1 and 3.2. Then

$$H^0(k,n) = \begin{cases} \bigcup_{r \text{ even}} K_r(A,B) & \text{if } k \text{ is odd,} \\ \bigcup_{r \text{ odd}} K_r(A,B) & \text{if } k \text{ is even,} \end{cases} \tag{6}$$

while

$$\overline{H^0(k,n)} = \begin{cases} \bigcup_{r \text{ odd}} K_r(A,B) = \bigcup_{r \text{ even}} K_r(B,A) & \text{if } k \text{ is odd,} \\ \bigcup_{r \text{ even}} K_r(A,B) & \text{if } k \text{ is even.} \end{cases} \tag{7}$$

Given a k-uniform hypergraph with $V(H) = V = A \cup B$, we denote by $E_r(A, B) := E_r$ the set of all edges of H intersecting A in precisely r elements, $r = 0, \ldots, k$. Whenever convenient, we will treat E_r as a sub-hypergraph of H. Note that E_r is $(r, k - r)$-bipartite. In the course of the proof we will often switch from the initial partition $V = A \cup B$ to a modified partition $V = A' \cup B'$. We will then use the shorthand notation $K'_r := K_r(A', B')$ and $E'_r := E_r(A', B')$. Note that if H ε-contains $H^0(k, n)$ or $\overline{H^0(k,n)}$ then, in view of (6) and (7) (see also Fig. 4), $|K_r \setminus E_r| < \varepsilon n^k$ for all r which appear in the range of the respective set union.

Our last preliminary result, unlike the three earlier, more general facts, applies directly to hypergraphs H satisfying the assumptions of Lemma 3.1.

Fact 4.4. *Assume that $V(H) := V = A \cup B$, $|A| \sim |B|$, $1 \leqslant r \leqslant k-1$, $|K_r(A, B) \setminus E_r(A, B)| < \varepsilon n^k$, and $\delta_{k-1}(H) \geqslant n/2 - O(1)$. Further, let S_A and S_B be the sets of vertices in A and B, respectively, which are 0.3-small in $E_r(A, B)$, and let $A' = (A \setminus S_A) \cup S_B$ and $B' = (B \setminus S_B) \cup S_A$. Then, for $n \geqslant n_0$,*

(a) *$|S_A| + |S_B| \leqslant \sqrt{\varepsilon} kn$, and*
(b) *for the new partition $V = A' \cup B'$, all vertices $v \in V$ are 0.2-large in $E_r(A', B')$.*

Proof. (a) Since $|K_r \setminus E_r| < \varepsilon n^k$, part (a) follows straight from Corollary 4.1 with $F = E_r$ and $c = 0.3$.

(b) We will first prove that every $v \in S_A$ is 0.6-large in E_{r+1}, while every $v \in S_B$ is 0.6-large in E_{r-1}.

Fix $v \in S_A$ and set $a = |A|$ and $b = |B|$. For any $a_1, \ldots, a_{r-1} \in A$ and $b_1, \ldots, b_{k-r-1} \in B$, the number of edges of H containing $v, a_1, \ldots, a_{r-1}, b_1, \ldots, b_{k-r-1}$ is

$$\deg_H(v, a_1, \ldots, a_{r-1}, b_1, \ldots, b_{k-r-1}) \geqslant \delta_{k-1}(H).$$

Adding over all choices of a's and b's, we obtain the inequality

$$\sum_{a\text{'s and } b\text{'s}} \deg_H(v, a_1, \ldots, a_{r-1}, b_1, \ldots, b_{k-r-1}) \geqslant \binom{a-1}{r-1}\binom{b}{k-r-1}\delta_{k-1}(H).$$

Recalling that $v \in S_A \subseteq A$, the above sum counts r times every edge of E_{r+1} containing v and it counts $k-r$ times every edge of E_r containing v. In other words, the inequality can be restated as

$$r\deg_{E_{r+1}}(v) + (k-r)\deg_{E_r}(v) \geqslant \binom{a-1}{r-1}\binom{b}{k-r-1}\delta_{k-1}(H).$$

By (5) this yields that

$$\begin{aligned}\deg_{E_{r+1}}(v) &\geqslant \frac{1}{r}\binom{a-1}{r-1}\binom{b}{k-r-1}\delta_{k-1}(H) - 0.3\frac{k-r}{r}\binom{a-1}{r-1}\binom{b}{k-r}\\ &\geqslant \deg_{K_{r+1}}(v)\left(\frac{n/2 - O(1)}{a-r} - 0.3\frac{b-k+r+1}{a-r}\right) \geqslant 0.6\deg_{K_{r+1}}(v),\end{aligned}$$

because $a \sim b$ and $n \geqslant n_0$. For $v \in S_B$ the proof is very similar and therefore omitted here.

Now we may complete the proof of Fact 4.4(b). Recall that $E'_r = E_r(A', B')$ and $K'_r = K_r(A', B')$ are defined with respect to the new partition $V = A' \cup B'$, described in the statement of Fact 4.4. Note that, by (a),

$$\left|\deg_{E'_r}(v) - \deg_{E_r}(v)\right| < \left(|S_A| + |S_B|\right)n^{k-2} < \sqrt{\varepsilon}kn^{k-1}.$$

Hence, for every 0.3-large vertex in E_r,

$$\deg_{E'_r}(v) \geqslant 0.3\deg_{K_r}(v) - \sqrt{\varepsilon}kn^{k-1} > 0.2\deg_{K'_r}(v).$$

On the other hand, for every $v \in S_A$ and sufficiently small $\varepsilon > 0$,

$$\deg_{E'_r}(v) \geqslant \deg_{E_{r+1}}(v) - \sqrt{\varepsilon}kn^{k-1} \geqslant 0.6\deg_{K_{r+1}}(v) - \sqrt{\varepsilon}kn^{k-1} > 0.2\deg_{K'_r}(v)$$

with an even bigger margin. The same is true for $v \in S_B$. □

4.2. The proof of Lemma 3.1

We will consider separately three cases:

Case 1: H ε-contains $H^0(k,n)$ and k is odd,
Case 2: H ε-contains $\overline{H^0(k,n)}$ and k is even, and
Case 3: H ε-contains $H^0(k,n)$ and k is even,

with the last case subdivided into two subcases, 3a and 3b, according to the parity of $k/2$. The reason for not treating the fourth case here is that for odd k $H^0(k,n)$ is (almost) self-complementary (compare (6) with (7) above), and thus, if H ε-contains $\overline{H^0(k,n)}$ then it, say, 2ε-contains $H^0(k,n)$.

Let H be a k-uniform hypergraph satisfying the assumptions of Lemma 3.1. We assume that $V(H) := V = A \cup B$, $A \cap B = \emptyset$, where $|A| = a(k,n)$ is determined in Definitions 3.1 and 3.2. Throughout this section, without further notice, we will use notation $a = |A|$, $b = |B|$, $a' = |B'|$, etc. Recall that $|a-b| = |2a(n,k) - n| \leqslant 2$. In our proofs, this initial partition will be modified slightly. The new partition $V = A' \cup B'$ will always be such that $|a' - b'| < \varepsilon' n$ for some $\varepsilon' = f(\varepsilon)$.

In all cases considered, for some values of r (depending on parity), we will have $|K_r \setminus E_r| < \varepsilon n^k$, and consequently $|E_r| = \Theta(n^k)$, as well as $|E'_r| = \Theta(n^k)$. Such indices r and the edges of such E_r's will be referred to as *typical*, while the others – *atypical*. In recognizing what is typical and what is not, the reader may be guided by Fig. 4 and the formulas (6) and (7). The indices appearing in the range of summation are the typical ones.

Recall that the absence of perfect matchings in the critical hypergraphs $H^0(k,n)$ defined in Section 3.1 is due to some parity problems. For instance, for k odd, $|A|$ is odd, while each edge of $H^0(k,n)$ intersects A in an even number of vertices. In the proof below, we will show that the degree condition (4) implies the existence of atypical edges. These, in turn, will serve as "parity breakers" to achieve required congruences.

Having all the preliminary results from Section 4.1 at hand, a general description of the construction of a perfect matching M in H goes as follows. Ideally, we would like to obtain a perfect matching right away by applying Fact 4.1 to $E_{k/2}$. But for this to work we would need to be very lucky: $k/2$ should be an integer and typical for the case (this is true only when $k=4\ell+2$ for some integer ℓ), we should have $a=b$, and no vertex should be ε-deficient in $E_{k/2}$.

We will show that the last two requests may be addressed by modifying the partition, and by matching the deficient vertices greedily, respectively. But there is nothing we can do about the parity of k. So, instead we will select two typical indices $0\leqslant r_1\neq r_2\leqslant k$ and apply Fact 4.1 twice: to suitably chosen sub-hypergraphs of E'_{r_1} and E'_{r_2}. For this to work, the "top sizes" of the two subgraphs should be multiples of r_1 and r_2, and the "bottom sizes" – multiples of $k-r_1$ and $k-r_2$, respectively.

In the template below, we first outline the general strategy. The first two steps are valid for all cases considered, while steps III and IV will have to be modified a little in Subcase 3b, because just then $k=4\ell+2$ and $r_1=k/2$ (see Remark 4.1 below for an explanation).

The template:

Let H be a k-uniform hypergraph satisfying the assumptions of Lemma 3.1. With some foresight fix suitable, *typical* r_1 and r_2, $1\leqslant r_1\leqslant k-1$, $0\leqslant r_2\leqslant k$.

I. Getting rid of small vertices in E_{r_1} by modifying the partition. By (2), (3) and (4), we have $\delta(H)=n/2+O(1)$, and by Definitions 3.1 and 3.2, we have $|A|\sim|B|$. Moreover, since H ε-contains $H^0(k,n)$ or $\overline{H^0(k,n)}$ and r_1 is typical, we also have $|K_{r_1}\setminus E_{r_1}|<\varepsilon n^k$. Thus, the assumptions of Fact 4.4 are satisfied for $r=r_1$.
Move all 0.3-small vertices of $E_{r_1}=E_{r_1}(A,B)$ to the other side. Due to Fact 4.4(b), for the new partition $V=A'\cup B'$, each $v\in V$ is 0.2-large in E'_{r_1}. Moreover, by Fact 4.4(a), $|a'-b'|\leqslant 2\sqrt{\varepsilon}kn$.

II. Fixing divisibility by removing small matching. Choose a matching M_1, consisting of at most two edges, so that the system of equations

$$r_1x+r_2y=a'',\qquad (k-r_1)x+(k-r_2)y=b'' \tag{8}$$

has an integer, non-negative solution (x,y), where $A''=A'\setminus V(M_1)$ and $B''=B'\setminus V(M_1)$. (In some cases, one of the edges of M_1 will need to be atypical; the existence of this atypical edge will follow from the degree condition (4).) Note that $A''\cup B''=V\setminus V(M_1)$ and

$$|a''-b''|\leqslant|a'-b'|+4k\leqslant 2\sqrt{\varepsilon}kn+4k.$$

III. Matching the deficient vertices first. Let N be the set of all vertices which are $\sqrt{\varepsilon}$-deficient in $E'_{r_1}[V\setminus V(M_1)]$ or in $E'_{r_2}[V\setminus V(M_1)]$. By Fact 4.2, $|N|\leqslant 2\sqrt{\varepsilon}kn$. In view of step I, all vertices are 0.1-large in $E'_{r_1}[V\setminus V(M_1)]$ (some degrees may have gotten a little smaller after removing $V(M_1)$).
We may thus apply Fact 4.3 with $F=E'_{r_1}[V\setminus V(M_1)]$, the above set N, and $\varepsilon:=2\sqrt{\varepsilon}k$, and construct in $E'_{r_1}[V\setminus V(M_1)]$ a matching $M_2=M(N)$ containing all vertices of N. Note that (8) still holds with A'', B'', x and y replaced, respectively, by $A'''=A''\setminus V(M_2)$, $B'''=B''\setminus V(M_2)$, $\tilde{x}=x-|M_2|$ and $\tilde{y}=y$, that is,

$$r_1\tilde{x}+r_2\tilde{y}=a''',\qquad (k-r_1)\tilde{x}+(k-r_2)\tilde{y}=b'''. \tag{9}$$

Note also that

$$|a'''-b'''|\leqslant|a''-b''|+2k|D|\leqslant 2\sqrt{\varepsilon}kn+4k+2k|N|\leqslant 7\sqrt{\varepsilon}k^2n. \tag{10}$$

IV. Obtaining a perfect matching M in H. So far we have constructed two disjoint matchings M_1 and M_2 in H, and a partition

$$V\setminus V(M_1\cup M_2)=A'''\cup B'''.$$

We know that no vertex is $\varepsilon^{1/3}$-deficient in $E'_{r_1}[A''' \cup B''']$ or in $E'_{r_2}[A''' \cup B''']$ (again, some degrees may have gotten smaller after removing $V(M_2)$). Also, $a''', b''', \tilde{x}$ and $\tilde{y}$ satisfy (9). All we need is a perfect matching in $H[A''' \cup B''']$.
In Cases 1, 2 and 3a, we will have $r_1, r_2 \neq k/2$. Then we further subdivide $A''' = A_1 \cup A_2$ and $B''' = B_1 \cup B_2$ so that

$$|A_1| = r_1\tilde{x}, \qquad |B_1| = (k - r_1)\tilde{x}, \qquad |A_2| = r_2\tilde{y}, \qquad |B_2| = (k - r_2)\tilde{y}.$$

It follows easily from (9) or (11) below that in this case $\tilde{x}, \tilde{y} \geqslant cn$ for some $c = c(k) > 0$ ($c = 1/(3k^2)$ would do). Thus, by Observation 1 there are no $\varepsilon^{1/3}/(kc)^{k-1}$-deficient vertices in $E_{r_1}(A_1, B_1)$ or $E_{r_2}(A_2, B_2)$. Hence, the assumptions of Fact 4.1 are satisfied for each of $E_{r_1}(A_1, B_1)$ and $E_{r_2}(A_2, B_2)$ with $\varepsilon := \varepsilon^{1/3}/(kc)^{k-1}$, and we obtain perfect matchings M_3 in $H[A_1 \cup B_1]$ and M_4 in $H[A_2 \cup B_2]$. Then $M = M_1 \cup M_2 \cup M_3 \cup M_4$ is a perfect matching of H.
Since in Subcase 3b we choose $r_1 = k/2$, we need to alter steps III and IV appropriately (cf. Remark 4.1 below).

After presenting the template, we give an insight into how the indices r_1 and r_2 will be selected. Solving (8), we obtain

$$x = \frac{1}{r_1 - r_2}\left(a'' - \frac{1}{k}(a'' + b'')r_2\right),$$
$$y = \frac{1}{k}(a'' + b'') - x = \frac{1}{r_2 - r_1}\left(a'' - \frac{1}{k}(a'' + b'')r_1\right). \tag{11}$$

Remembering that k divides $a'' + b''$, one can easily check that the integrality of the solution (x, y) is equivalent to the congruence

$$a'' - \frac{1}{k}(a'' + b'')r_2 \equiv 0 \pmod{r_1 - r_2}. \tag{12}$$

The easiest way to satisfy (12) is to set $r_2 = 0$ and request that $a'' \equiv 0 \pmod{r_1}$. But this works only when $r = 0$ is a typical index. Another good choice, reducing (12) to a parity question, is when $r_1 - r_2 = \pm 2$. We will use both these ideas in the actual proof below.

Remark 4.1. When $k = 4\ell + 2$, then $k/2$ is a typical index of the critical construction H^0, and a natural choice is $r_1 = k/2$. Then, however, by (11) we have $y = (a'' - b'')/(2r_2 - k) < \varepsilon'' n$. This means that the sub-hypergraph $E_{r_2}(A_2, B_2)$ considered in step IV, may be too small to apply to it Observation 1, and consequently Fact 4.1. Fortunately, there is an alternative approach: in step IV apply Fact 4.3 to an arbitrary set N of $\tilde{y}$ vertices, all of which are 0.1-large in $E'_{r_2}[A''' \cup B''']$, and then apply Fact 4.1 to $E'_{k/2}[A_1 \cup B_1]$, where $A_1 = A''' \setminus V(M(N))$ and $B_1 = B''' \setminus V(M(N))$. Then, of course, in step III we need only to match the deficient vertices of $E'_{r_1}[V \setminus V(M_1)]$ (see Subcase 3b below for details).

Now we are ready to consider each case separately. Set $H^0 = H^0(k, n)$ and $\delta^0 = \delta^0(k, n) = \delta_{k-1}(H^0)$. Let H be a k-uniform hypergraph with vertex set $V = A \cup B$, satisfying the assumptions of Lemma 3.1. Recall that $E_r(A, B) := E_r$ is the set of all edges of H intersecting A in precisely r elements, $r = 0, \ldots, k$.

Case 1 – H ε-contains $H^0(k, n)$ and k is odd

Recall that in this case $H^0 \subset_\varepsilon H$ and thus we have $|K_r \setminus E_r| < \varepsilon n^k$ for each even r. Take $r_1 = k - 1$ and $r_2 = 0$. The congruence (12) reduces to the demand that $a'' \equiv 0 \pmod{k - 1}$.

I. Let S_A and S_B be the sets of vertices in A and B, respectively, which are 0.3-small in E_{k-1}, and let $A' = (A \setminus S_A) \cup S_B$ and $B' = (B \setminus S_B) \cup S_A$. By Fact 4.4, all vertices are 0.2-large in $E'_{k-1} := E'_{k-1}(A', B')$. Moreover, $a' + b' = n$ and $|a' - b'| \leqslant 2\sqrt{\varepsilon}kn$.

Preparing for step II, we prove now the following fact.

Fact 4.5. *If a' is odd then $E'_1 \cup E'_{k-2} \neq \emptyset$.*

Proof. We first show that

$$a' + 2 - k \leqslant \delta^0 \quad \text{or} \quad b' + 1 - k \leqslant \delta^0. \tag{13}$$

Suppose that $a' + 1 - k \geqslant \delta^0$ and $b' - k \geqslant \delta^0$. Adding up sidewise, this yields that $\delta^0 \leqslant n/2 + 1/2 - k$. Comparing with (2), we see that this is a contradiction except when $n = 4m + 1$ and $a' + 1 - k = b' - k = \delta^0$. But then $a' = b' - 1$ which implies that $a' = 2m$ – a contradiction.

Hence (13) holds. If $a' + 2 - k \leqslant \delta^0$ then consider a $(k-1)$-tuple of vertices $a_1, \dots, a_{k-2} \in A'$ and $b_{k-1} \in B'$. Since $\delta_{k-1}(H) \geqslant \delta^0 + 1$, it must have a neighbor $b_k \in B'$. The edge $\{a_1, \dots, a_{k-2}, b_{k-1}, b_k\}$ belongs to E'_{k-2}. Similarly, if $b' + 1 - k \leqslant \delta^0$, take a $(k-1)$-tuple of vertices $b_1, \dots, b_{k-2}, b_{k-1} \in B'$. Again, there must be a neighbor, this time in A', forcing $E'_1 \neq \emptyset$. □

II. In this step we construct a matching M_1 of size $|M_1| \in \{0, 1, 2\}$ such that for $A'' = A' \setminus V(M_1)$ and $B'' = B' \setminus V(M_1)$ we will have $a'' \equiv 0 \pmod{k-1}$. Let $s \equiv a' \pmod{k-1}$. Suppose that $s \neq 0$. For even s, we will use an edge from E'_s, and there are plenty of those (all even s are typical in this case). However, for odd s we need an atypical edge, provided by Fact 4.5 (note that since s is odd, a' must be odd, since $k-1$ is even).

Thus, depending on s, we define matching M_1 as follows.

(1) If s is even, set $M_1 = \{e\}$, where e is an edge from E'_s. (For $s = 0$, set $M_1 = \emptyset$.)
(2) If s is odd and $E'_1 \neq \emptyset$, set $M_1 = \{e_1, e_2\}$, where e_1 is an edge from E'_1 and e_2 is an edge from E'_{s-1}. (For $s = 1$, we do not need e_2).
(3) If s is odd and $E'_{k-2} \neq \emptyset$, set $M_1 = \{e_1, e_2\}$, where e_1 is an edge from E'_{k-2} and e_2 is an edge from E'_{s+1}. (For $s = k-2$, we do not need e_2).

It can be easily checked that in each case we do have $a'' \equiv 0 \pmod{k-1}$. This means that (12) holds, which in turn implies that the system (8) has a positive, integer solution (x, y).

III. Let N be the set of vertices which are $\sqrt{\varepsilon}$-deficient in $E'_{k-1}[V \setminus V(M_1)]$ or in $E'_0[V \setminus V(M_1)]$. As in the template, we apply Fact 4.3 with $r = k-1$, and $\varepsilon := 2\sqrt{\varepsilon}k$, and construct in $E'_{k-1}[V \setminus V(M_1)]$ a matching $M_2 = M(N)$ containing all vertices of N. With $A''' = A'' \setminus V(M_2)$ and $B''' = B'' \setminus V(M_2)$, we see that

$$a''' = a'' - (k-1)|M_2| \equiv 0 \pmod{k-1}.$$

Thus, the solution to the system (9), $\tilde{x} = a'''/(k-1)$ and $\tilde{y} = (b''' - \tilde{x})/k$, is integer.

IV. First, fix a subset B_1 of B''' of size $|B_1| = a'''/(k-1)$ and apply Fact 4.1 with $r = k-1$ to the sub-hypergraph $E'_{k-1}[A''' \cup B_1]$ (here $A_1 = A'''$ because $r_2 = 0$). As a result, we obtain a perfect matching M_3 in $E'_{k-1}[A''' \cup B_1]$. Next, apply Fact 4.1 with $r = 0$ to the sub-hypergraph $E'_0[B''' \setminus B_1]$ (here $A_2 = \emptyset$ and $B_2 = B''' \setminus B_1$), obtaining a perfect matching M_4 in $E'_0[B''' \setminus B_1]$. The matching $M = M_1 \cup M_2 \cup M_3 \cup M_4$ is a perfect matching of H.

Case 2 – H ε-contains $\overline{H^0(k,n)}$ and k is even

Let us recall that the vertex set of $\overline{H^0(k,n)}$ is split into A and B, where $|A| = a(k,n)$, and the edge set consists of all sets of k vertices which intersect A (and thus B) in an even number of elements. Note that the degree condition in Lemma 3.1 is stated with respect to the original critical hypergraph $H^0(k,n)$, and not with respect to its complement $\overline{H^0(k,n)}$.

The proof in this case is very similar to that in the previous case. Again, all even indices are typical. Here we choose $r_1 = k-2$ and $r_2 = 0$. The congruence (12) reduces to $a'' \equiv 0 \pmod{k-2}$.

I. As in Case 1, we obtain a partition $V = A' \cup B'$ such that all vertices are 0.2-large in E'_{k-2}.

Fact 4.6. $E'_1 \cup E'_{k-1} \neq \emptyset$.

Proof. Suppose that $a' - k \geqslant \delta^0$ and $b' - k \geqslant \delta^0$. Adding up sidewise, this yields that $\delta^0 \leqslant n/2 - k$. Comparing with (3), we see that this is a contradiction. Thus, either $a' + 1 - k \leqslant \delta^0$ or $b' + 1 - k \leqslant \delta^0$.

If $a' + 1 - k \leqslant \delta^0$, consider a $(k-1)$-tuple of vertices $a_1, \dots, a_{k-1} \in A'$. Since $\delta_{k-1}(H) \geqslant \delta^0 + 1$, there is a vertex $b \in B'$ such that $\{a_1, \dots, a_{k-1}, b\} \in E'_{k-1}$. By symmetry, it follows that if $b' + 1 - k \leqslant \delta^0$ then $E'_1 \neq \emptyset$. $\square$

II. We are going to construct a matching M_1 of size $|M_1| \in \{0, 1, 2\}$ such that for $A'' = A' \setminus V(M_1)$ and $B'' = B' \setminus V(M_1)$ we will have $a'' \equiv 0 \pmod{k-2}$. Let $s \equiv a' \pmod{k-2}$. Suppose that $s \neq 0$. For even s, we will use an edge from E'_s. For odd s we need an atypical edge provided by Fact 4.6.

Depending on s, we define matching M_1 as follows.

(1) If s is even, set $M_1 = \{e\}$, where e is an edge from E'_s. (For $s = 0$, set $M_1 = \emptyset$.)
(2) If s is odd, set $M_1 = \{e_1, e_2\}$, where e_1 is an edge from $E'_1 \cup E'_{k-1}$ and e_2 is an edge from E'_{s-1}. (For $s = 1$, we do not need e_2.)

It can be easily checked that in each case we do have $a'' \equiv 0 \pmod{k-2}$. This means that (12) holds, which in turn implies that the system (8) has a positive, integer solution (x, y).

III., IV. As in Case 1, but with $r_1 = k-2$ and $r_2 = 0$.

Case 3 – H ε-contains $H^0(k,n)$ and k is even

We will consider separately two subcases: when $k/2$ is even and when $k/2$ is odd.

Subcase 3a: $k = 4\ell$ for some integer $\ell \geqslant 1$.

Set $r_1 = k/2 + 1$, $r_2 = k/2 - 1$ and note that the congruence condition (12) becomes equivalent to the requirement that $\frac{1}{2}(a'' - b'') + \frac{1}{k}(a'' + b'')$ is even.

I. Follow step I of the template with $r_1 = k/2 + 1$. Note that $a' - b'$ is even (because $a' + b' = n$ is even) and assume by symmetry that $a' \geqslant b'$. Again, we have $a' - b' \leqslant 2\sqrt{\varepsilon} kn$.

II. Our goal is to find a matching M_1 such that with $A'' = A' \setminus V(M_1)$, $B'' = B' \setminus V(M_1)$, either both, $\frac{1}{2}(a'' - b'')$ and $\frac{1}{k}(a'' + b'')$, are even, or both are odd. In other words, we aim at one of the following two "desired" situations: for some integer s and m, either

(i) $a'' - b'' = 4s$ and $a'' + b'' = 2mk$,

or

(ii) $a'' - b'' = 4s + 2$ and $a'' + b'' = (2m+1)k$.

If (i) or (ii) is already satisfied by a' and b', we set $M_1 = \emptyset$. Suppose that a' and b' satisfy neither (i) nor (ii). Then they must satisfy one of the following:

(iii) $a' - b' = 4s + 2$ and $a' + b' = 2mk$,

or

(iv) $a' - b' = 4s$ and $a' + b' = (2m+1)k$.

In these two cases we set $M_1 = \{e\}$, where $e \in E'_2$. Note that then a'' and b'' do satisfy either (i) or (ii). It remains to show that $E'_2 \neq \emptyset$.

Fact 4.7. *If* (iii) *or* (iv) *holds with $a' \geqslant b'$ then $E'_2 \neq \emptyset$. (If $a' \leqslant b'$ then $E'_{k-2} \neq \emptyset$.)*

Proof. If $b' + 2 - k \leqslant \delta^0$, then we are done because $\delta_{k-1}(H) \geqslant \delta_0 + 1$. So, suppose that $\delta^0 \leqslant b' + 1 - k$. If (iii) holds then $b' \leqslant n/2 - 1$, so $\delta^0 \leqslant n/2 - k$, a contradiction with (3). If (iv) holds then we only have $b' \leqslant n/2$ and $\delta^0 \leqslant n/2 + 1 - k$, but $n/k = (a' + b')/k = 2m + 1$, so, by (3) we have $\delta^0 = n/2 + 2 - k$ and, again, we arrive at a contradiction. $\square$

III. Follow step III of the template with $r_1 = k/2 + 1$ and $r_2 = k/2 - 1$. With $A''' = A'' \setminus V(M_2)$ and $B''' = B'' \setminus V(M_2)$, we see that a''' and b''' still satisfy one of the desired conditions, (i) or (ii) (every edge included to M_2 makes $a''' - b'''$ smaller by 2 and at the same time it makes $a''' + b'''$ smaller by k). Hence, (9) has an integer solution, $\tilde{x} = m + s$, $\tilde{y} = m - s$.

IV. If a''', b''' satisfy (i), then $a''' = mk + 2s$ and $b''' = mk - 2s$. Partition arbitrarily $A''' = A_1 \cup A_2$ and $B''' = B_1 \cup B_2$, where $|A_1| = (m + s)(k/2 + 1)$, $|B_1| = (m + s)(k/2 - 1)$, $|A_2| = (m - s)(k/2 - 1)$, and $|B_2| = (m - s)(k/2 + 1)$. These partitions are possible because (9) holds.

Apply Fact 4.1 with $r = k/2 + 1$ to the sub-hypergraph $E'_{k/2+1}[A_1 \cup B_1]$. As a result, we obtain a perfect matching M_3 of size $m + s$. Next, apply Fact 4.1 with $r = k/2 - 1$ to the sub-hypergraph $E'_{k/2-1}[A_2 \cup B_2]$, obtaining a perfect matching M_4 of size $m - s$. The matching $M = M_1 \cup M_2 \cup M_3 \cup M_4$ is a perfect matching of H. If a''', b''' satisfy (ii), we proceed similarly, except that now $|M_3| = m + s + 1$ and $|M_4| = m - s$.

Subcase 3b: $k = 4\ell + 2$ for some integer $\ell \geqslant 1$.

Set $r_1 = k/2$ and $r_2 = k/2 \pm 2$, with the sign at 2 depending on the outcome of step I. The congruence condition (12) becomes equivalent to the requirement that $\frac{1}{2}(a'' - b'')$ is even.

I. Follow step I of the template with $r_1 = k/2$. Note that $a' - b'$ is even and assume by symmetry that $a' \geqslant b'$. Again, $a' - b' \leqslant 2\sqrt{\varepsilon} kn$.

Set $r_2 = k/2 + 2$ (In the other, symmetric case when $a' \leqslant b'$, we take $r_2 = k/2 - 2$.) For step II we need the following fact.

Fact 4.8. *If $a' > b'$ then $E'_2 \neq \emptyset$. (If $a' < b'$ then $E'_{k-2} \neq \emptyset$.)*

Proof. We will first show that $b' + 2 - k \leqslant \delta^0$. Suppose then that $b' + 1 - k \geqslant \delta^0$. But $b' \leqslant n/2 - 1$, so $\delta^0 \leqslant n/2 - k$, a contradiction with (3). So, $b' + 2 - k \leqslant \delta^0$. Since $\delta_{k-1}(H) \geqslant \delta^0 + 1$, any $(k-1)$-tuple of vertices $b_1, \dots, b_{k-2} \in B'$, $a_{k-1} \in A'$, must have a neighbor in A', and hence this will complete the proof. □

II. In order to satisfy (12), we just need the number $\frac{1}{2}(a' - b')$ to be even. If it is not, that is, if $a' - b' = 4s + 2$ for some s, we get to the even case by removing form H (an atypical) edge $e \in E'_2$. Indeed, then we have

$$a'' - b'' = (a' - 2) - (b' - k + 2) = a' - b' - 4 + k = 4s + 2 - 4 + 4\ell + 2 = 0 \pmod 4. \tag{14}$$

In Fact 4.8 above we have shown that $E'_2 \neq \emptyset$.

Now we construct M_1. If $a' - b' = 4s$, set $M_1 = \emptyset$. If $a' - b' = 4s + 2$, let $M_1 = \{e\}$, where $e \in E'_2$. Set $A'' = A' \setminus V(M_1)$ and $B'' = B' \setminus V(M_1)$ and note that, by (14) $0 \leqslant a'' - b'' = 4t \leqslant 2\sqrt{\varepsilon} kn + k - 4$, where $t = s + 2\ell$.

Since $r_1 = k/2$ and thus in the solution (x, y) of (8) the value of $y = (a'' - b'')/4 = t$ is very small, the next two steps are slightly different than in the template.

III. Follow step III of the template but only for $r_1 = k/2$, i.e., construct a matching M_2 in $E'_{k/2}[V \setminus V(M_1)]$ containing all $\sqrt{\varepsilon}$-deficient vertices in $E'_{k/2}[V \setminus V(M_1)]$. Set $A''' = A'' \setminus V(M_2)$ and $B''' = B'' \setminus V(M_2)$. Since each edge of M_2 intersects both sets A'' and B'' in $k/2$ vertices, it is obvious that still $a''' - b''' = a'' - b'' = 4t$.

IV. Recall that $t \leqslant \frac{1}{2}\sqrt{\varepsilon} kn + \ell$ and choose any set N of t vertices which are 0.1-large in $E'_{k/2+2}[A''' \cup B''']$. This is possible because, due to Corollary 4.1, most vertices are 0.1-large. Let $M_3 = M(N)$ be a matching guaranteed by Fact 4.3. It consists of precisely t edges of $E'_{k/2+2}[A''' \cup B''']$. Note that for $A_1 = A''' \setminus V(M_3)$ and $B_1 = B''' \setminus V(M_3)$, we have $|A_1| = |B_1|$.

Finally, apply Fact 4.1 with $r = k/2$ to the sub-hypergraph $E'_{k/2}[A_1 \cup B_1]$. As a result, we obtain a perfect matching M_4. The matching $M = M_1 \cup M_2 \cup M_3 \cup M_4$ is a perfect matching of H.

This concludes the proof of Lemma 3.1.

5. Away from the critical construction

In this section we complete the proof of Theorem 3.1 by showing Lemma 3.2. Recall that $H^0(k,n)$ is the critical graph and that $\delta^0(k,n) = \delta_{k-1}(H^0(k,n))$ (see Section 3).

As in Section 4, set $H^0 := H^0(k,n)$ and $\delta^0 = \delta^0(k,n)$. For the proof of Lemma 3.2 we will need the following properties of hypergraphs H which, at the same time, do not ε-contain H^0 and do not ε-contain $\overline{H^0}$ (cf. Definition 3.3).

Given k (not necessarily disjoint) sets $N_i \subseteq V(H)$, $i = 1,\dots,k$, denote by $E_H(N_1,\dots,N_k)$ the set of ordered k-tuples of *distinct* vertices $(v_1,\dots,v_k)$ such that $v_i \in N_i$, $i = 1,\dots,k$, and $\{v_1,\dots,v_k\} \in H$. Set

$$e_H(N_1,\dots,N_k) = \left|E_H(N_1,\dots,N_k)\right|.$$

Note that the same edge of H may be counted more than once and that

$$e_H(N_1,\dots,N_k) = e_H(N_{\sigma(1)},\dots,N_{\sigma(k)}) \tag{15}$$

for any permutation σ of the index set $[k]$.

Claim 5.1. *For every $\varepsilon > 0$ there exists n_0 such that if the assumptions of Lemma 3.2 hold, that is,*

- *H is a k-uniform hypergraph on $n > n_0$ vertices, n divisible by $k \geqslant 3$,*
- *$\delta_{k-1}(H) \geqslant (1/2 - 1/\log n)n$, and*
- *$H^0(k,n) \not\subset_\varepsilon H$ and $\overline{H^0(k,n)} \not\subset_\varepsilon H$,*

then at least one of the following conditions holds.

(i) *For all $N_1,\dots,N_k \subseteq V(H)$ with $|N_i| \geqslant (1/2 - 1/\log n)n$, we have*

$$e_H(N_1,\dots,N_k) \geqslant \frac{n^k}{\log^3 n}.$$

(ii) *Setting $\Lambda = \{(v_1,\dots,v_{k-1})\colon \deg_H(v_1,\dots,v_{k-1}) > (1/2 + 2/\log n)n\}$, we have*

$$|\Lambda| \geqslant \frac{n^{k-1}}{\log n}.$$

Proof. Set $\gamma = 1/\log n$ for convenience and suppose that neither (i) nor (ii) holds. Our goal is to find inside H a sub-hypergraph G which ε-contains either H^0 or $\overline{H^0}$. This will be a contradiction with the third assumption of Claim 5.1.

Since (i) does not hold, there exist sets $N_1,\dots,N_k$ with $|N_i| \geqslant (1/2 - \gamma)n$ and

$$e_H(N_1,\dots,N_k) < \frac{n^k}{\log^3 n}. \tag{16}$$

Fact 5.1. *If* (16) *holds, then*

(a) *for all $i = 1,\dots,k$, $|N_i| < (1/2 + 2\gamma)n$,*
(b) *for all $1 \leqslant i < j \leqslant k$,*

$$|N_i \cap N_j| < \gamma n \quad \textit{or} \quad |N_i \cup N_j| < \left(\frac{1}{2} + 2\gamma\right)n. \tag{17}$$

Proof. (a) Suppose that there exists i such that $|N_i| \geqslant (1/2 + 2\gamma)n$. Without loss of generality, let us assume for convenience that $i = k$. Then, for all choices of $v_j \in N_j$, $j = 1,\dots,k-1$,

$$\left|N_H(v_1,\dots,v_{k-1}) \cap N_k\right| \geqslant \gamma n,$$

so

$$e_H(N_1,\dots,N_k) \geqslant |N_1|\big(|N_2|-1\big)\cdots\big(|N_{k-1}|-(k-2)\big)\big(\gamma n-(k-1)\big)$$
$$=\Omega\big(\gamma n^k\big)=\Omega\bigg(\frac{n^k}{\log n}\bigg),$$

a contradiction with (16) for large n.

(b) Suppose there are $i \neq j$ such that

$$|N_i \cap N_j| \geqslant \gamma n \quad \text{and} \quad |N_i \cup N_j| \geqslant (1/2+2\gamma)n.$$

Without loss of generality, let us assume for convenience that $i=k-1$ and $j=k$. Then, for all choices of $v_\ell \in N_\ell$, $\ell=1,\dots,k-2$, and $v \in N_{k-1}\cap N_k$,

$$\big|N_H(v_1,\dots,v_{k-2},v)\cap(N_{k-1}\cup N_k)\big| \geqslant \gamma n.$$

Consider $w \in N_H(v_1,\dots,v_{k-2},v)\cap(N_{k-1}\cup N_k)$. If $w\in N_k$ then $(v_1,\dots,v_{k-2},v,w)\in E_H(N_1,\dots,N_k)$, while if $w \in N_{k-1}$ then $(v_1,\dots,v_{k-2},w,v)\in E_H(N_1,\dots,N_k)$. Consequently,

$$e_H(N_1,\dots,N_k) \geqslant |N_1|\big(|N_2|-1\big)\cdots\big(|N_{k-2}|-(k-3)\big)\big(\gamma n-(k-2)\big)\big(\gamma n-(k-1)\big)$$
$$=\Omega\big(\gamma^2 n^k\big)=\Omega\bigg(\frac{n^k}{\log^2 n}\bigg),$$

again a contradiction with (16) for large n. □

Since $\gamma = 1/\log n = o(1)$ when $n\to\infty$, Fact 5.1(a) implies that for all $i=1,\dots,k$,

$$|N_i| = n/2 + o(n), \tag{18}$$

while Fact 5.1(b) implies that for all $1\leqslant i<j\leqslant k$,

$$|N_i\cap N_j| = o(n) \quad \text{or} \quad |N_i\cup N_j| < \frac{1}{2}n+o(n). \tag{19}$$

These two facts together mean that there exists j_0, $0 \leqslant j_0 \leqslant k$, so that some j_0 sets among $N_1,\dots,N_k$ are essentially the same and almost disjoint from the remaining sets, which then are also essentially the same. This brings us closer to our goal of constructing a sub-hypergraph G of H which ε-contains either H^0 or $\overline{H^0}$.

Without loss of generality we will assume that the first j_0 sets, $N_1,\dots,N_{j_0}$ are essentially the same and almost disjoint from $N_{j_0+1},\dots,N_k$. Recall that $V(H^0)=A\cup B$, where $|A|=a(k,n)=n/2+O(1)$ is given in Definitions 3.1 and 3.2. Owing to (18) and (19), there exists a set $A\subset V$ of size $|A|=a(k,n)$ such that, setting $B=V\setminus A$, for every $i=1,\dots,j_0$,

$$|A \triangle N_i| = o(n) \tag{20}$$

while for $i=j_0+1,\dots,k$,

$$|B \triangle N_i| = o(n). \tag{21}$$

Let for each $j=0,\dots,k$, $(jA,(k-j)B)$ stand for the sequence of j copies of the set A followed by $k-j$ copies of B. With this notation, (16), (20) and (21) imply that

$$e_H\big(j_0A,(k-j_0)B\big)=o\big(n^k\big). \tag{22}$$

We will next show the following fact.

Fact 5.2.

(a) *For every* $j = 1, \ldots, k$,

$$e_H\big(jA, (k-j)B\big) = o\big(n^k\big) \quad \Rightarrow \quad e_H\big((j-1)A, (k-j+1)B\big) = (n/2)^k + o\big(n^k\big).$$

(b) *For every* $j = 2, \ldots, k$,

$$e_H\big((j-1)A, (k-j+1)B\big) = (n/2)^k + o\big(n^k\big) \quad \Rightarrow \quad e_H\big((j-2)A, (k-j+2)B\big) = o\big(n^k\big).$$

Proof. First note that for all $0 \leqslant j \leqslant k$,

$$e_H\big(jA, (k-j)B\big) \leqslant |A|^j |B|^{k-j} = (n/2)^k + o\big(n^k\big). \tag{23}$$

(a) Observe that, using (15) and setting $V = V(H)$ and $(x)_t = x(x-1)\cdots(x-t+1)$,

$$\begin{aligned} e_H\big((j-1)A, (k-j+1)B\big) &= e_H\big((j-1)A, (k-j)B, V\big) - e_H\big((j-1)A, (k-j)B, A\big) \\ &\geqslant \big(|A|\big)_{j-1}\big(|B|\big)_{k-j}\big(\delta_{k-1}(H) - (k-1)\big) - e_H\big(jA, (k-j)B\big) \\ &\geqslant (n/2)^k - o\big(n^k\big). \end{aligned}$$

Thus, part (a) follows by (23).

(b) Recall that we have assumed that neither condition (i) nor (ii) of Claim 5.1 holds. In fact, below we will use only a weaker consequence of negating (ii), namely that $|\Lambda| = o(n^{k-1})$ (see Claim 5.1 for the definition of Λ).

Let $\Phi = (A^{j-2} \times B^{k-j+1}) \setminus \Lambda$. Note that for all $(v_1, \ldots, v_{k-1}) \in \Phi$, we have

$$\deg_H(v_1, \ldots, v_{k-1}) \leqslant (1/2 + 2\gamma)n. \tag{24}$$

Also, given $W \subset V$, let

$$\deg_H(v_1, \ldots, v_{k-1}; W) = \big|N_H(v_1, \ldots, v_{k-1}) \cap W\big|$$

be the number of collective neighbors of $\{v_1, \ldots, v_{k-1}\}$ in H, which belong to W. Then, because $|\Lambda| = o(n^{k-1})$, $\deg_H(v_1, \ldots, v_{k-1}) \leqslant n$, and using (15) we have

$$e_H\big((j-1)A, (k-j+1)B\big) = \sum_{(v_1,\ldots,v_{k-1}) \in \Phi} \deg_H(v_1, \ldots, v_{k-1}; A) + o\big(n^k\big)$$

and

$$e_H\big((j-2)A, (k-j+2)B\big) = \sum_{(v_1,\ldots,v_{k-1}) \in \Phi} \deg_H(v_1, \ldots, v_{k-1}; B) + o\big(n^k\big).$$

Let us denote the sums appearing on the right-hand side's above, respectively, by $\sum_A$ and $\sum_B$. Then, by (24),

$$\sum_A + \sum_B = \sum_{(v_1,\ldots,v_{k-1}) \in \Phi} \deg_H(v_1, \ldots, v_{k-1}) \leqslant \big(n/2 + O(1)\big)^{k-1}\big(n/2 + o(n)\big).$$

Since by our assumption $\sum_A = (n/2)^k + o(n^k)$, we infer that $\sum_B = o(n^k)$, and consequently $e_H((j-2)A, (k-j+2)B) = o(n^k)$. $\square$

Fact 5.2 together with (22) imply that for every $0 \leqslant j \leqslant j_0$,

$$e_H\big(jA, (k-j)B\big) = \begin{cases} o(n^k) & \text{if } j \equiv j_0 \pmod 2, \\ (n/2)^k + o(n^k) & \text{if } j \not\equiv j_0 \pmod 2. \end{cases} \tag{25}$$

Interchanging the roles of A and B, (25) then follows for all $j = 0, \ldots, k$. In particular, this implies that

$$\big|E_j(A, B) \bigtriangleup K_j(A, B)\big| = o\big(n^k\big) \quad \text{for all } j \not\equiv j_0 \pmod 2,$$

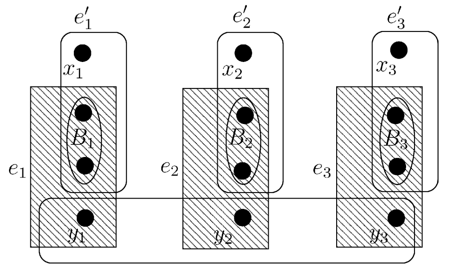

Fig. 5. An S-absorbing k-matching ($k=3$).

which in view of (6) and (7) means that

$$G = \bigcup_{j \not\equiv j_0 \pmod 2} E_j(A,B) \subseteq H$$

ε-contains H^0 or $\overline{H^0}$, depending on the parities of k and j_0. Since $G \subseteq H$, this is a contradiction with our assumption that H does not ε-contain either of these graphs, and thus completes the proof of Claim 5.1. □

Having proved Claim 5.1, in order to finish the proof of Lemma 3.2, we will show that hypergraphs H satisfying the degree condition $\delta_{k-1}(H) \geqslant (1/2 - 1/\log n)n$ and one of the conditions (i) or (ii) listed in Claim 5.1 contain a perfect matching.

Claim 5.2. *There exists n_0 such that if H is a k-uniform hypergraph on $n > n_0$ vertices, where n is divisible by k, with $\delta_{k-1}(H) \geqslant (1/2 - 1/\log n)n$ and at least one of the conditions* (i) *and* (ii) *of Claim* 5.1 *holds, then H has a perfect matching.*

Proof of Lemma 3.2. Lemma 3.2 follows from Claims 5.1 and 5.2. To see this, recall that Lemma 3.2 and Claim 5.1 have the same assumptions. Claim 5.1 asserts that under these assumptions one of the conditions (i) and (ii) holds. On the other hand, Claim 5.2 guarantees the existence of a perfect matching in H under either of these conditions. Thus, Lemma 3.2 is proved. □

It remains to prove Claim 5.2. Its proof relies on the absorbing technique shown already in the proof of Proposition 2.1. We will first describe two "absorbing devices," one for each of the conditions, (i) and (ii). Then we will show that for each set S of k vertices there are in H many absorbing devices (Fact 5.3), and using this, that there is a universal "absorbing" matching M' (Fact 5.4, proved via Proposition 5.1). Finally, we will provide a short proof of Claim 5.2 based on Fact 5.4 and Proposition 2.1.

We define two similar types of "absorbing devices" for a given set S of k vertices. One of them will consist of a k-matching, disjoint from S, for which there exists a $(k+1)$-matching covering all its vertices together with the set S; the other one will be a $(k+1)$-matching, for which there exists a $(k+2)$-matching covering all its vertices together with the set S.

Definition 5.1 *(k-edge absorbing device).* Given a set $S = \{x_1, \ldots, x_k\}$ of k vertices of H, we call a k-matching $\{e_1, \ldots, e_k\}$ in H *S-absorbing* if there is in H a $(k+1)$-matching $\{e_1', \ldots, e_k', f\}$ such that (see Fig. 5)

- $e_i' \cap e_j = \emptyset$ for all $i \neq j$,
- $e_i' \setminus e_i = \{x_i\}$ and $\{y_i\} := e_i \setminus e_i'$ for all $i = 1, \ldots, k$,
- $f = \{y_1, \ldots, y_k\}$.

The other absorbing device is very similar.

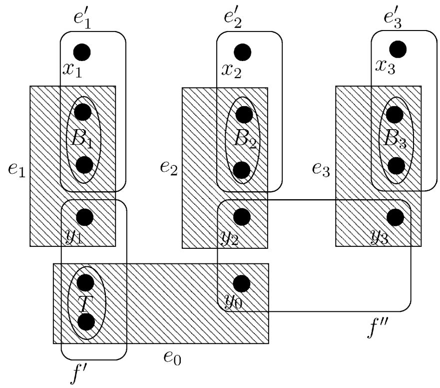

Fig. 6. An S-absorbing $(k+1)$-matching $(k=3)$.

Definition 5.2 *($(k+1)$-edge absorbing device).* Given a set $S=\{x_1,\dots,x_k\}$ of k vertices of H, we call a $(k+1)$-matching $\{e_0,e_1,\dots,e_k\}$ in H *S-absorbing* if there is in H a $(k+2)$-matching $\{e'_1,\dots,e'_k,f',f''\}$ such that (see Fig. 6)

- $e'_i\cap e_j=\emptyset$ for all $i\neq j$,
- $e'_i\setminus e_i=\{x_i\}$ and $\{y_i\}:=e_i\setminus e'_i$ for all $i=1,\dots,k$,
- $f'\cap e_1=\{y_1\}=f'\setminus e_0$, $f''=\{y_0,y_2,\dots,y_k\}$, where $y_0:=e_0\setminus f'$.

We will use these devices in the following context. Given a set S and a matching M' such that $V(M')\cap S=\emptyset$, if M' contains an S-absorbing k-matching then we will modify M' by swapping $e_1,\dots,e_k$ with $e'_1,\dots,e'_k$ and f. This way, the resulting matching $(M'\setminus\{e_1,\dots,e_k\})\cup\{e'_1,\dots,e'_k,f\}$ will have the vertex set $V(M')\cup S$. Similarly, if M' contains an S-absorbing $(k+1)$-matching then M' will "absorb" S by swapping $e_0,e_1,\dots,e_k$ with $e'_1,\dots,e'_k,f'$ and f''.

Next, we show that if at least one of the conditions (i) and (ii) of Claim 5.1 holds, then for each set S there are many absorbing devices in H.

Fact 5.3. *There exists n_0 such that the following holds. Let H be a k-uniform hypergraph with $n\geqslant n_0$ vertices and $\delta_{k-1}(H)\geqslant(1/2-1/\log n)n$, and let $S=\{x_1,\dots,x_k\}$ be a set of k vertices of H.*

(i) *If condition* (i) *of Claim* 5.1 *holds, then the number of S-absorbing k-matchings in H is $\Omega(n^{k^2}/\log^3 n)$.*
(ii) *If condition* (ii) *of Claim* 5.1 *holds, then the number of S-absorbing $(k+1)$-matchings in H is $\Omega(n^{k^2+k}/\log^3 n)$.*

Proof. In the proofs below, the reader should be guided by Figs. 5 and 6.

(i) Given $S=\{x_1,\dots,x_k\}$, for every $i=1,\dots,k$, there are $\Theta(n^{k-1})$ sets B_i such that $e'_i=\{x_i\}\cup B_i\in H$. Consequently, there are $\Theta(n^{k(k-1)})$ choices of (disjoint) sets $B_1,\dots,B_k$ forming the first $k-1$ vertices of each of $e_1,\dots,e_k$. Let $N_i=N_H(B_i)$, $i=1,\dots,k$. Since $|N_i|>(1/2-1/\log n)n$ and we assume (i) of Claim 5.1, there are $\Omega(n^k/\log^3 n)$ choices of the edge $f=\{y_1,\dots,y_k\}$ such that the sets $B_i\cup\{y_i\}$, $i=1,\dots,k$, form k disjoint edges $e_1,\dots,e_k$. Hence, altogether there are

$$\Omega\left(n^{k(k-1)}\times n^k/\log^3 n\right)=\Omega\left(n^{k^2}/\log^3 n\right)$$

choices of S-absorbing k-matchings, as claimed.

(ii) As in case (i), there are $\Theta(n^{k(k-1)})$ choices of (disjoint) sets $B_1,\dots,B_k$ forming with the vertices of S the edges $e'_1,\dots,e'_k$, and, at the same time, being the first $k-1$ vertices of $e_1,\dots,e_k$, respectively. For $i=2,\dots,k$, we choose $y_i\in N_H(B_i)$, each in at least $(1/2-1/\log n)n-O(1)$ ways. This way we have constructed the edges $e_i = B_i \cup \{y_i\}$, $i = 2, \ldots, k$. Next, we select a $(k-1)$-element sequence of vertices $T \in \Lambda$, which is disjoint from

$$S \cup B_1 \cup \cdots \cup B_k \cup \{y_2, \ldots, y_k\}.$$

Since (ii) of Claim 5.1 holds there are $\Theta(n^{k-1}/\log n)$ choices of T. By the definition of Λ, we have $|N_H(B_1) \cap N_H(T)| \geqslant n/\log n$ as well as $|N_H(\{y_2, \ldots, y_k\}) \cap N_H(T)| \geqslant n/\log n$. Consequently, we can select $y_1 \in N_H(B_1) \cap N_H(T)$ and $y_0 \in N_H(\{y_2, \ldots, y_k\}) \cap N_H(T)$, each in at least $n/\log n - O(1)$ ways. This yields $f' = T \cup \{y_1\}$ and $e_0 = T \cup \{y_0\}$.

Summarizing, we have chosen B's, y's, T, y_0 and y_1, forming an S-absorbing $(k+1)$-matching, in

$$\Omega\left(n^{k(k-1)} \times n^{k-1} \times \frac{n^{k-1}}{\log n} \times \left(\frac{n}{\log n}\right)^2\right) = \Omega\left(\frac{n^{k^2+k}}{\log^3 n}\right)$$

ways, as claimed. □

Our last preparatory result establishes the existence of a universal "absorbing" matching M'.

Fact 5.4. *There exists n_0 such that the following holds. Let H be a k-uniform hypergraph with $n \geqslant n_0$ vertices and $\delta_{k-1}(H) \geqslant (1/2 - 1/\log n)n$.*

(i) *If condition* (i) *of Claim* 5.1 *holds, then there exists a matching M' in H of size $|M'| = O(\log^4 n)$ such that for every k-element set S of vertices of H, there is at least one S-absorbing k-matching contained in M'.*
(ii) *If condition* (ii) *of Claim* 5.1 *holds, then there exists a matching M' in H of size $|M'| = O(\log^4 n)$ such that for every k-element set S of the vertices of H, there is at least one S-absorbing $(k+1)$-matching contained in M'.*

Both parts of Fact 5.4 will follow from a more general result.

Proposition 5.1. *Let ℓ, k, and d be positive integers, $m = m(n) \leqslant n^d$ be a polynomial function of n, and $\alpha = \alpha(n)$ satisfy*

$$\frac{\alpha\sqrt{n}}{\log n} \to \infty$$

as $n \to \infty$. Then there exists n_0 such that if

- *H is a k-uniform hypergraph with $n \geqslant n_0$ and*
- *$\mathcal{F}_1, \ldots, \mathcal{F}_m$ are families of ℓ-matchings in H of sizes $|\mathcal{F}_i| \geqslant \alpha n^{\ell k}$, $i = 1, \ldots, m$,*

then there exists a matching M' in H of size $|M'| = O((\log n)/\alpha)$ such that for every $i = 1, \ldots, m$,

$$\binom{M'}{\ell} \cap \mathcal{F}_i \neq \emptyset.$$

Proof of Fact 5.4. Let H be a k-uniform hypergraph on n vertices, satisfying the degree condition from Fact 5.4. Let $m = \binom{n}{k}$ and let $S_1, \ldots, S_m$ be all k-element subsets of $V(H)$. Suppose that condition (i) of Claim 5.1 holds. Then, for each set S_i consider the family $\mathcal{F}_i$ of all S_i-absorbing k-matchings. By Fact 5.3(i), these $\mathcal{F}_i$, $i = 1, \ldots, m$, satisfy the assumptions of Proposition 5.1 with $\ell = k$, $m = \binom{n}{k}$, and $\alpha = c'/\log^3 n$ for a suitable constant $c' > 0$. If condition (ii) of Claim 5.1 holds, we proceed similarly: for each set S_i consider the family $\mathcal{F}_i$ of all S_i-absorbing $(k+1)$-matchings. By Fact 5.3(ii), these $\mathcal{F}_i$, $i = 1, \ldots, m$, satisfy the assumptions of Proposition 5.1 with $\ell = k+1$, $m = \binom{n}{k}$, and $\alpha = c''/\log^3 n$ for a suitable constant $c'' > 0$. In each case, by Proposition 5.1 there is a matching M' satisfying the conclusion of Fact 5.4. □

Our proof of Proposition 5.1 below is probabilistic. It is similar to that of Fact 2.3, but instead of Chernoff's bound it is based on Janson's inequality.

Proof of Proposition 5.1. Select a random subset M' of H, where each edge is chosen independently with probability

$$p = C(\log n)\alpha^{-1}n^{-k}, \tag{26}$$

where $C > d2^{\ell}$. Then, the expected size of M' is at most

$$\binom{n}{k}p < \frac{n^k p}{k!} = \frac{C\log n}{\alpha k!}$$

and the expected number of intersecting pairs of edges in M' is

$$O\left(n^{2k-1}p^2\right) = O\left(\frac{\log^2 n}{\alpha^2 n}\right) = o(1),$$

by our assumption on α. Hence, by Markov's inequality, M' is a matching of size at most $C\log n/\alpha$ with probability at least $1 - 1/k! - o(1)$.

To prove the intersection property of M', we will rely on a special case of Janson's inequality. Let $X_i = |\binom{M'}{\ell} \cap \mathcal{F}_i|$, $i = 1, \dots, m$. Further, for each $M \in \mathcal{F}_i$, let $I(M) = 1$ if $M \subseteq M'$ and $I(M) = 0$ otherwise. For a fixed index i, we will show that $\mathbb{P}(X_i = 0) \leqslant \exp\{-C\log n/2^{\ell}\}$. We have

$$\mathbb{E}(X_i) = |\mathcal{F}_i|p^{\ell} \geqslant \alpha n^{\ell k}p^{\ell},$$

while

$$\begin{aligned}
&\sum\sum\left\{\mathbb{E}\big(I(M_1)I(M_2)\big)\colon\ M_1, M_2 \in \mathcal{F}_i,\ M_1 \cap M_2 \neq \emptyset\right\} \\
&\quad= \sum_{t=1}^{\ell}\sum\sum\left\{p^{2\ell-t}\colon\ M_1, M_2 \in \mathcal{F}_i,\ |M_2 \cap M_1| = t\right\} \leqslant \sum_{t=1}^{\ell}\sum_{M_1\in\mathcal{F}_i}\binom{\ell}{t}n^{k(\ell-t)}p^{2\ell-t} \\
&\quad= \sum_{t=1}^{\ell}\binom{\ell}{t}|\mathcal{F}_i|n^{k(\ell-t)}p^{2\ell-t} = \mathbb{E}(X_i)\sum_{t=1}^{\ell}\binom{\ell}{t}n^{k(\ell-t)}p^{\ell-t} \leqslant 2^{\ell}\mathbb{E}(X_i)n^{k(\ell-1)}p^{\ell-1},
\end{aligned}$$

because $n^k p > 1$.

Hence, by Janson's inequality [2, Theorem 2.18(ii)] and (26),

$$\begin{aligned}
\mathbb{P}(X_i = 0) &\leqslant \exp\left\{-\frac{(\mathbb{E}(X_i))^2}{\sum\sum_{M_1\cap M_2\neq\emptyset}\mathbb{E}(I(M_1)I(M_2))}\right\} \\
&\leqslant \exp\left\{-\alpha n^k p/2^{\ell}\right\} = \exp\left\{-C\log n/2^{\ell}\right\},
\end{aligned}$$

and so

$$\mathbb{P}(X_i = 0 \text{ for some } i = 1, \dots, m) \leqslant m\exp\left\{-C\log n/2^{\ell}\right\} = o(1),$$

since $C > d2^{\ell}$ and $m \leqslant n^d$. Thus, with probability at least $1 - 1/k! - o(1)$, M' is a matching of size at most $C\log n/\alpha$ and such that $X_i = |\binom{M'}{\ell} \cap \mathcal{F}_i| \geqslant 1$, $i = 1, \dots, m$. Consequently, there exists a matching M' as required. □

We are finally ready for a short proof of Claim 5.2.

Proof of Claim 5.2. Let n_0 be determined by Fact 5.4, and let $n > n_0$ be divisible by k. Further, let H be a k-uniform hypergraph on n with $\delta_{k-1}(H) \geqslant (1/2 - 1/\log n)n$. We first assume that H satisfies condition (i) of Claim 5.1. The proof proceeds in three steps.

1. Let M' be a matching in H as described in Fact 5.4(i), and let $H' = H - V(M')$. Note that, with $n' := |V(H')|$,

$$\delta_{k-1}(H') \geqslant \delta_{k-1}(H) - O\left(\log^4 n\right) > \frac{2n}{5} > \frac{2n'}{5}.$$

2. Remove from H' an arbitrary vertex v and observe that

$$\delta_{k-1}(H'-v) \geqslant \delta_{k-1}(H') - 1 > \frac{2n'}{5} - 1.$$

Since $n'-1$ is not divisible by k, by Proposition 2.1, the threshold for the existence of a matching of size $\lfloor (n'-1)/k \rfloor = n'/k - 1$ is $t(k, n'-1) \leqslant n/k + O(\log n)$. Hence,

$$\delta_{k-1}(H'-v) > t(k, n'-1)$$

and, consequently, there is in $H'-v$ a matching M'' of size $n'/k - 1$.
3. Let $S = V(H') \setminus V(M'')$, $|S| = k$. Take an S-absorbing k-matching $\{e_1, \ldots, e_k\}$ contained in M' and guaranteed by Fact 5.4(i), together with an accompanying $(k+1)$-matching $\{e'_1, \ldots, e'_k, f\}$ satisfying Definition 5.1. Absorb S into M' by setting $M'_S = (M' \setminus \{e_1, \ldots, e_k\}) \cup \{e'_1, \ldots, e'_k, f\}$. Then $V(M'_S) = V(M') \cup S$ and consequently $M'_S \cup M''$ is a perfect matching of H.

If H satisfies condition (ii) of Claim 5.1, the proof proceeds mutatis mutandis. □

References

[1] R. Aharoni, A. Georgakopoulos, Ph. Sprüssel, Perfect matchings in r-partite r-graphs, European J. Combin. 30 (1) (2009) 39–42.
[2] S. Janson, T. Łuczak, A. Ruciński, Random Graphs, John Wiley and Sons, New York, 2000.
[3] G.Y. Katona, H.A. Kierstead, Hamiltonian chains in hypergraphs, J. Graph Theory 30 (1999) 205–212.
[4] D. Kühn, D. Osthus, Matchings in hypergraphs of large minimum degree, J. Graph Theory 51 (2006) 269–280.
[5] V. Rödl, A. Ruciński, E. Szemerédi, Perfect matchings in uniform hypergraphs with large minimum degree, European J. Combin. 27 (2006) 1333–1349.
[6] V. Rödl, A. Ruciński, E. Szemerédi, A Dirac-type theorem for 3-uniform hypergraphs, Combin. Probab. Comput. 15 (1–2) (2006) 229–251.
[7] V. Rödl, A. Ruciński, E. Szemerédi, An approximative Dirac-type theorem for k-uniform hypergraphs, Combinatorica 28 (2) (2008) 229–260.