Sum of Degrees of Vertices Theorem

Theorem (Sum of Degrees of Vertices Theorem)

Suppose a graph has n vertices with degrees $d_1, d_2, d_3, \ldots, d_n$. Add together all degrees to get a new number $d_1 + d_2 + d_3 + \ldots + d_n = D_v$. Then $D_v = 2e$.

In words, for any graph the sum of the degrees of the vertices equals twice the number of edges.

Stated in a slightly different way, $D_v = 2e$ says that D_v is <u>ALWAYS</u> an even number.

Example (Using the Sum of Degrees of Vertices Theorem)

It is impossible to make a graph with $v = 6$ where the vertices have degrees 1, 2, 2, 3, 3, 4. This is because the sum of the degrees D_v is

$$D_v = 1 + 2 + 2 + 3 + 3 + 4 = 15$$

D_v is always an even number but 15 is odd!

Using the Sum of the Degrees of Vertices Formula 1

Example (Using the Sum of the Degrees of Vertices Formula 1)

Consider the following scenarios:

- A graph has 4 vertices with degrees 0, 0, 0, and 0. What does this graph look like?

- A graph has 1 vertex with degree 2. What does this graph look like?

- A graph has 4 vertices with degrees 2, 3, 3, and 4. How many edges are there?

- A graph has 4 vertices with degrees 2, 2, 2, and 4. Can you say what this graph looks like?

Using the Sum of the Degrees of Vertices Formula 2

Example (Using the Sum of the Degrees of Vertices Formula 2)

Consider the following scenarios:

- Is it possible to have a graph with vertices of degrees: 1 and 1?

- Is it possible to have a graph with vertices of degrees: 1 and 2?

- Is it possible to have a graph with vertices of degrees: 1, 1, 2, 3?

- Is it possible to have a graph with vertices of degrees: 1, 1, 2, 3, 3?
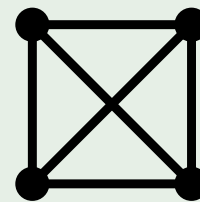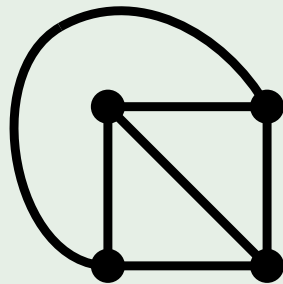
Planar Graphs

Definition (Planar Graph)

A graph is **Planar** if can be drawn in such a way that its edges do not cross.

To determine if a graph is planar we have to consider isomorphic versions of the graph.

Example (Using Isomorphisms to Make Planar)

The graph on the left is definitely planar (edges do not cross). Even though the graph to the right does not appear to be planar,**it is because it is isomorphic to the graph to the left!**

Faces of a Planar Graphs

Definition (Faces of a Planar Graphs)

In any planar graph, drawn with no intersections, the edges divide the planes into different regions.

The regions enclosed by the planar graph are called **interior faces** of the graph.

The region surrounding the planar graph is called the **exterior (or infinite) face** of the graph.

When we say **faces** of the graph we mean **<u>BOTH</u>** the interior AND the exterior faces. We usually denote the number of faces of a planar graph by f.

BEFORE YOU COUNT FACES, IT IS VERY IMPORTANT TO FIRST DRAW A PLANAR GRAPH SO THAT NO EDGES CROSS!
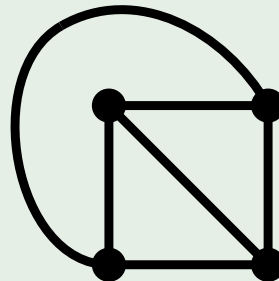
Counting Faces of a Planar Graphs

Example (Counting Faces of a Planar Graphs)

It is really tempting to try to count the faces in the graph below:
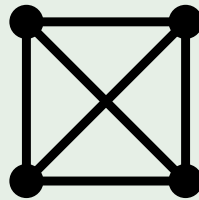
Remember, the idea of faces only applies to Planar Graphs drawn with no edge crossings. Instead we need to redraw the graph to get an isomorphic version:
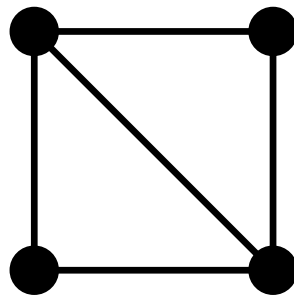

In the bottom version it is easy to see there are *3 Interior Faces and 1 Exterior Face for a total of 4 Faces*.

Degree of a Face

Definition (Degree of a Face)

For a planar graph drawn without edges crossing, the number of edges bordering a particular face is called the **degree of the face**.
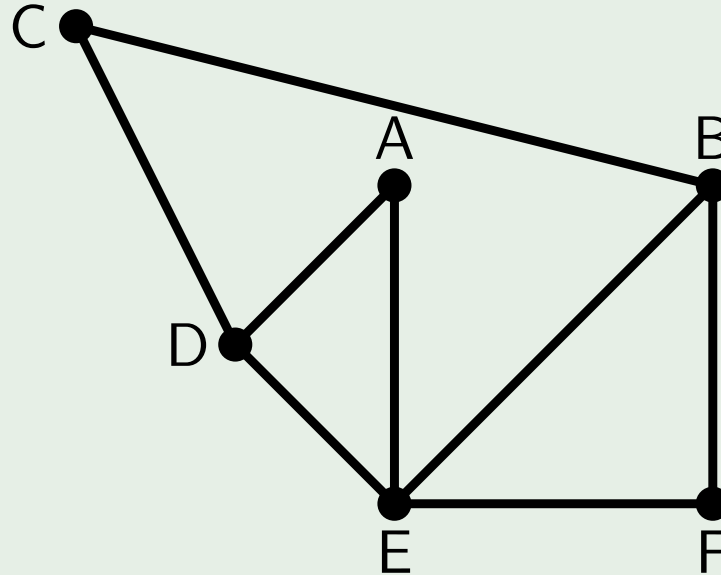
In the graph below

each of the two interior faces have degree 3. The infinite (exterior) face has degree 4. Adding up degrees gives a result that is curiously similar to one before.

Theorem (Sum of the Degrees For Faces)

In any planar graph, the sum of the degrees of all faces is equal to twice the number of edges. In symbols, $D_f = 2e$.
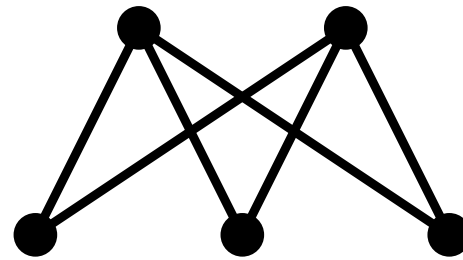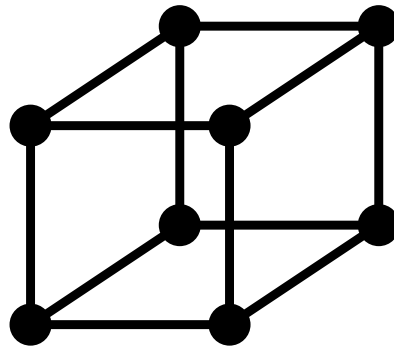
Counting Faces and Degrees 1

- How many faces does this graph have?

- What is the degree of each face?

- List v, e, and f for this graph.

Making Graphs Look Planar

Here are a couple more graphs that don't look planar at first. Can you count the faces?
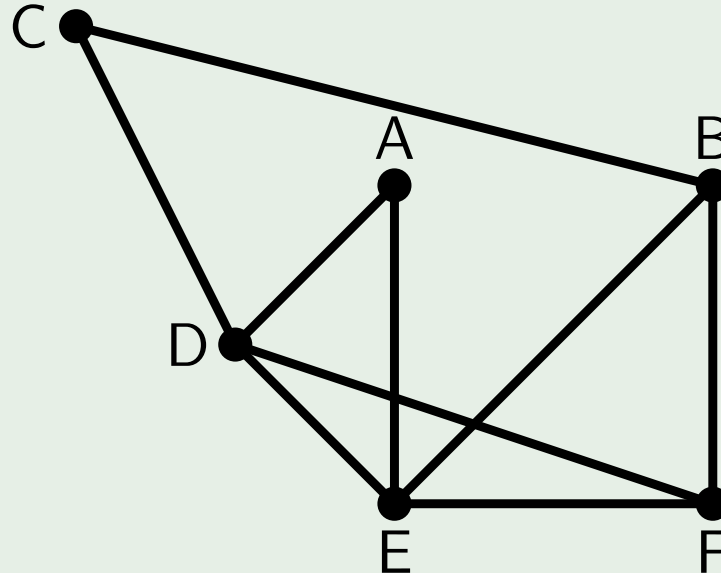

	Left Graph (Cube)	Right Graph ($K_{3,2}$)
Vertices v	8	5
Edges e	12	6
Faces f	6	3

These numbers are giving more evidence of some big ideas with planar graphs.
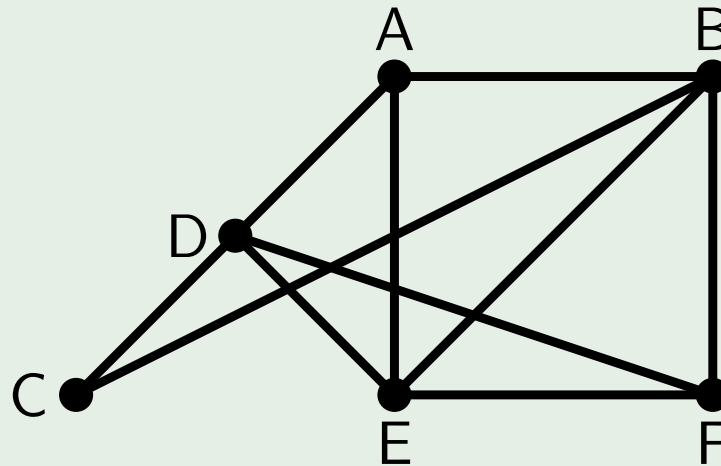
Example (Counting Faces and Degrees 2)

- How many faces does this graph have?

- What is the degree of each face?

- List v, e, and f for this graph.

Example (Counting Faces and Degrees 3)

- How many faces does this graph have?

- What is the degree of each face?

- List v, e, and f for this graph.

Euler's Formula

Theorem (Euler's Formula)

*Take any connected planar graph drawn with no intersecting edges.
Let v be the number of vertices in the graph.*

Let e be the number of edges in the graph.

Let f be the number of faces in the graph.

$$\text{Then } v - e + f = 2.$$

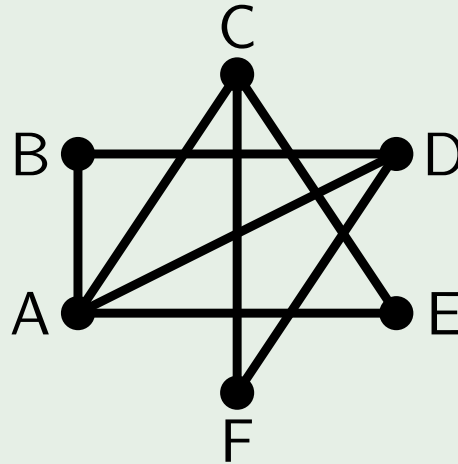Check the planar graphs on the previous slide:

- Cube $\qquad\qquad v - e + f = 8 - 12 + 6 = 2$

- $K_{3,2} \qquad\qquad v - e + f = 5 - 6 + 3 = 2$

Let's see what Euler's Formula can do for us!

Using Euler's Formula 1

- What is v?

- What is e?

- If the graph is planar, what must f be?

- Redraw the graph with vertices/edges moved around so the graph appears planar.

Useful Facts for Planar Graphs

We can combine the following theorems to answer questions about planar graphs.

Theorem (Sum of the Degrees For Vertices)

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

along with what we just looked at

Theorem (Sum of the Degrees For Faces)

In any planar graph, the sum of the degrees of all faces is equal to twice the number of edges.

along with Euler's Formula

Theorem (Euler's Formula)

For a connected planar graph with vertices v, edges e, and faces f, the following must hold:

$$v - e + f = 2.$$

Using Euler's Formula 2

Example (Using Euler's Formula 2)

- A connected planar graph has 8 vertices and 12 edges. How many faces are there?

- A connected planar graph has 6 vertices and 4 faces. How many edges are there?

- A connected planar graph has 8 vertices with degrees: 1, 1, 2, 2, 3, 3, 4, 4. How many edges are there?

- A connected planar graph has 8 vertices with degrees: 1, 1, 2, 2, 3, 3, 4, 4. How many faces are there?

Using Euler's Formula 3

Example (Using Euler's Formula 3)

- A connected planar graph has 6 faces and 12 edges. How many vertices are there?

- A connected planar graph has 4 faces with degrees: 3, 3, 4, 4. How many edges are there?

- A connected planar graph has 4 faces with degrees: 3, 3, 4, 4. How many vertices are there?

- Draw a connected planar graph with 4 faces with degrees: 3, 3, 4, 4. Is there only one such isomorphic graph?

Using Euler's Formula 4

Example (Using Euler's Formula 4)
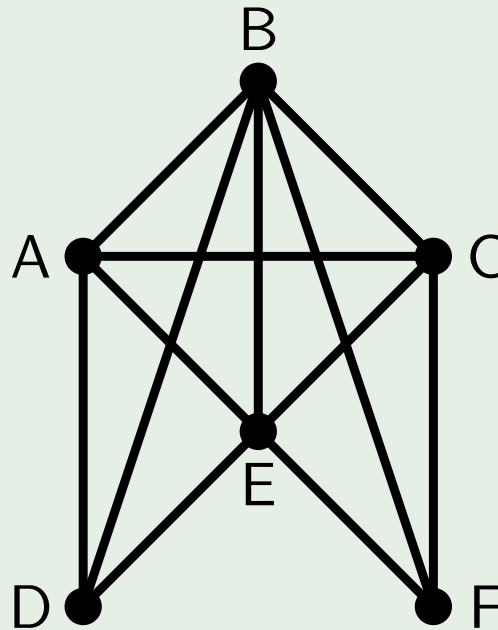
- What is v? What is e?

- Though it doesn't look it, this is graph is planar. What is f?