UNMIXED d-UNIFORM r-PARTITE HYPERGRAPHS

REZA JAFARPOUR-GOLZARI AND RASHID ZAARE-NAHANDI

Abstract. In this paper, we characterize all unmixed d-uniform r-partite hypergraphs under a certain condition. Also we give a necessary condition for unmixedness in d-uniform hypergraphs with a perfect matching of size n. Finally we give a sufficient condition for unmixednes in d-uniform hypergraphs with a perfect matching.

1. Introduction

Unmixedness is one of the most important concepts in theory of graphs and hypergraphs with nice and interesting algebraic and geometric interpretations (for instance see [4], [6], [7], [9], [10], [13]). According to this, characterization of special classes of unmixed graphs has been noteworthy in recent years. G. Ravindra in [8] and R. H. Villarreal in [11] have characterized all unmixed bipartite graphs independently. H. Haghighi in [2] has given a characterization for unmixed tripartite graphs under a certain condition, and recently R. Jafarpour-Golzari and R. Zaare-Nahandi in [5] have generalized Haghighi's result for unmixed r-partite graphs. On characterization of unmixed r-partite hypergraphs, almost no study has been done. Only in [6] a classification of a very special class of unmixed multipartite hypergraphs has been provided.

In this paper we give a characterization of all unmixed d-uniform r-partite hypergraphs under a certain condition which we name it $(**)$. Also we give necessary or sufficient conditions for unmixedness in more general classes of d-uniform hypergraphs (Propositions 3.6, 3.8).

MSC(2010): Primary: 5E40; Secondary: 5C65.

Keywords: r-partite hypergraph, d-uniform hypergraph, minimal vertex cover, independent set, unmixed, perfect matching.

.

2. **Preliminaries**

In the sequel, we use [12] and [1] for terminology and notations on graphs and hypergraphs respectively.

Let $G = (V, E)$ be a simple finite graph. For $x, y \in V$, $x \sim y$ means that x and y are adjacent. A subset M of V is said to be independent if for every $x, y \in M$, $x \nsim y$. A vertex cover for G is a subset C of V such that every edge of G, intersects C. A vertex cover C is minimal whenever there is no any pure subset of C which is a vertex cover. G is called is called unmixed if all minimal vertex cover of G have the same number of elements. A subset Q of V is said to be a clique if for every two distinct vertices $x, y \in Q$, $x \sim y$.

A hypergraph $\mathcal{H}$ on a finite nonempty set V is a set of nonempty subsets of V such that $\bigcup_{e\in\mathcal{H}} e = V$. The elements of V are called vertices and each element of $\mathcal{H}$ is said a hyperedge. We denote by $V(\mathcal{H})$ and $E(\mathcal{H})$, the sets of vertices and hyperedges of $\mathcal{H}$ respectively. A hypergraph is said to be simple hypergraph or clutter if non of its two distinct hyperedges contains another. The hypergraph $\mathcal{H}$ is called d-uniform (or d-graph), if all its hyperedges have the same cardinality d.

Definition 2.1. *An r-partite ($r \geq 2$) hypergraph $\mathcal{H}$, is a hypergraph which $V(\mathcal{H})$ can be partitioned to r subsets such that for every two vertices x, y in one part, x, y do not lie in any hyperedge. Such a partition of $V(\mathcal{H})$ is called an r-partition of $\mathcal{H}$. If $r = 2, 3$, the r-partite hypergraph is said to be bipartite and tripartite respectivily.*

In the hypergraph $\mathcal{H}$, two vertices x, y are said to be adjacent if there is a hyperedge containing x and y. We say that a hyperedge e is adjacent with a vertex x if $x \in e$. For a vertex x of $\mathcal{H}$, the neighborhood of x, denoted by $N(x)$, is the set of all vertices which are adjacent to x.

A subset M of $V(\mathcal{H})$ is called independent if it dose not contain any hyperedge. An independent set M of $\mathcal{H}$ is said to be maximal whenever it is not strictly contained in any other independent set. A subset C of $V(\mathcal{H})$ is called a vertex cover, if every hyperedge of $\mathcal{H}$ intersects it. A vertex cover is said to be minimal if there is no any pure subset of it which is also a vertex cover. It is clear that every maximal independent set is complement of a minimal vertex cover and vice versa.

Definition 2.2. *The hypergraph $\mathcal{H}$ is said to be unmixed if all minimal vertex covers of $\mathcal{H}$ have the same cardinality.*

A matching in a hypergraph $\mathcal{H}$ is a set of hyperedges which are disjoint pairwise. A perfect matching is a matching such that every vertex of $\mathcal{H}$ lies in at least one of its elements.

Let $\{1, \ldots, n\}$ is denoted by $[n]$. A simplicial complex on $[n]$ is a set Δ of subsets of $[n]$ such that (a) $\{x\} \in \Delta$, for every $x \in [n]$, (b) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. Each element of Δ is said to be a face. Dimension of the face F, denoted by dim F, is defined as $|F| - 1$ and dimension of Δ is the maximum of dimentions of its faces.

Let Δ be a simplicial complex on $[n]$ and S be a nonempty set of subsets of $[n]$ such that $\bigcup_{N \in S} N = [n]$. The simplicial complex generated by S is the set of all subsets of elements of S.

For a simplicial complex Δ or a hypergraph $\mathcal{H}$ on $[n]$, and for $r \geq 0$, the r-skeleton of Δ or $\mathcal{H}$, is the set of all faces of Δ whose dimension is at most r or all subsets of hyperedges of $\mathcal{H}$ with cardinality not exceeding $r + 1$, respectively.

Definition 2.3. *Let $\mathcal{H}$ be a d-uniform ($d \geq 2$) hypergraph. A $(d-1)$-subset of a hyperedge is called a submaximal edge, and the set of all submaximal edges is denoted by $SE(\mathcal{H})$.*

For $\mathfrak{e} \in SE(\mathcal{H})$, the set $\{v \in V(\mathcal{H}) \mid \mathfrak{e} \cup \{v\} \in E(\mathcal{H})\}$, is denoted by $N(\mathfrak{e})$. If $v \in N(\mathfrak{e})$, we write $\mathfrak{e} \sim v$.

In a d-uniform hypergraph $\mathcal{H}$, a clique is a subset W of $V(\mathcal{H})$ such that every its subset of size d, is a hyperedge in $\mathcal{H}$.

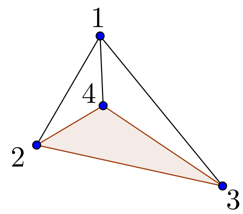
3. **Unmixed hypergraphs**

Let $\mathcal{H}$ is a d-uniform r-partite hypergraph with $2 \leq d \leq r$. We say that $\mathcal{H}$ satisfies the condition $(**)$ for $r \geq 2$, if $\mathcal{H}$ can be partitioned to r parts $V_i = \{x_{1i}, \cdots, x_{ni}\}$, $1 \leq i \leq r$, such that $\{x_{j1}, x_{j2} \ldots, x_{jr}\}$ is a clique for every $1 \leq j \leq n$.

The authors in [5] have presented a necessary and sufficient condition for unmixedness of an r-partite graph which satisfies the following condition $(*)$ for $r \geq 2$.
We say a graph G satisfies the condition $(*)$ for an integer $r \geq 2$, if G can be partitioned to r parts $V_i = \{x_{1i}, \ldots, x_{ni}\}$, $1 \leq i \leq r$, such that for all $1 \leq j \leq n$, $\{x_{j1}, \ldots, x_{jr}\}$ is a clique.

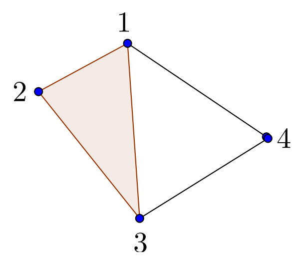
Let $\mathcal{H}$ be a d-uniform r-partite hypergraph $(2 \leq d \leq r)$ on $[n]$ which satisfies the condition $(**)$ for $r \geq 2$. Then the 1-skeleton of $\mathcal{H}$ is an r-partite graph which satisfies the condition $(*)$ for r. But in general, the unmixedness of a hypergraph and its 1-skeleton, are two independent facts, as the following example exhibits.

Example 3.1. The following clutter is not unmixed while its 1-skeleton is unmixed as a graph.



Note that the sets $\{2,4,3\}$ and $\{1,4\}$ are two minimal vertex covers with different sizes for the hypergraph.

Conversely, the following clutter is unmixed but its 1-skeleton is not.



Note that the sets $\{1,3\}$ and $\{2,3,4\}$ are two minimal vertex covers with different sizes for the 1-skeleton.

This gives us a motivation for finding a necessary and sufficient condition under which a d-uniform r-partite $(2 \leq d \leq r)$ hypergraph satisfying the condition $(**)$ for r, is unmixed.

First we prove a lemma.

Lemma 3.2. *Let $\mathcal{H}$ be a d-uniform r-partite $(2 \leq d \leq r)$ hypergraph which satisfies the condition $(**)$ for r. If $\mathcal{H}$ is unmixed, then every minimal vertex cover of $\mathcal{H}$ contains exactly $r-d+1$ elements of each clique $\{x_{j1}, x_{j2} \ldots, x_{jr}\}$.*

Proof. Let C be a minimal vertex cover of $\mathcal{H}$. For every $1 \leq q \leq n$, C contains at least $r-d+1$ vertices of the clique $\{x_{q1}, x_{q2} \ldots, x_{qr}\}$, because if C contains at most $r-d$ vertices of that clique, it dose not cover hyperedges on remaining vertices. Therefore a vertex cover must contain at least $n(r-d+1)$ vertices. On the other hand, the set $\bigcup_{i=1}^{r-d+1} V_i$ is a minimal vertex cover of $\mathcal{H}$ with $n(r-d+1)$ vertices. This completes the proof. □

Now we present the main theorem of this paper.

Theorem 3.3. *Let $\mathcal{H}$ be a d-uniform r-partite ($2 \leq d \leq r$) hypergraph which satisfies the condition (**) for r. Then $\mathcal{H}$ is unmixed if and only if the following condition holds.*
For every $1 \leq q \leq n$, if $\mathfrak{e}_1, \mathfrak{e}_2 \ldots, \mathfrak{e}_{r-d+2}$ are submaximal edges such that

$$\mathfrak{e}_1 \sim x_{qi_1}, \mathfrak{e}_2 \sim x_{qi_2}, \ldots, \mathfrak{e}_{r-d+2} \sim x_{qi_{r-d+2}}$$

where $i_1, i_2 \ldots, i_{r-d+2}$ are distinct, then the set

$$\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \ldots \cup \mathfrak{e}_{r-d+2}$$

is not independent.

Proof. Let $\mathcal{H}$ be unmixed. We show that the mentioned condition holds. Suppose in contrary

$$\mathfrak{e}_1 \sim x_{qi_1}, \mathfrak{e}_2 \sim x_{qi_2}, \ldots, \mathfrak{e}_{r-d+2} \sim x_{qi_{r-d+2}}$$

where $i_1, i_2 \ldots, i_{r-d+2}$ are distinct but the set

$$F = \mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \ldots \cup \mathfrak{e}_{r-d+2}$$

is independent. Therefore there is a maximal independent set M containing F. Since M is a maximal independent set, $C := V(\mathcal{H}) \backslash M$ is a minimal vertex cover of $\mathcal{H}$ which contains no any element of F. Since C is a vertex cover of $\mathcal{H}$, C contains the vertices $x_{qi_1}, x_{qi_2}, \ldots, x_{qi_{r-d+2}}$. But by Lemma 3.2, C contains exactly $r-d+1$ vertices of every clique, a contradiction.

Conversely, let the mentioned condition holds. We show that $\mathcal{H}$ is unmixed. It is enough to show that every minimal vertex cover of $\mathcal{H}$ contains exactly $r-d+1$ vertices of each clique $\{x_{q1}, x_{q2} \ldots, x_{qr}\}$. Let C be an arbitrary minimal vertex cover and $1 \leq q \leq n$. Then C intersects the set $\{x_{q1}, x_{q2} \ldots, x_{qr}\}$ in at least $r-d+1$ elements. Let C intersects the mentioned clique in at least $r-d+2$ elements. Without loss of generality, we assume that this elements are $x_{q1}, x_{q2} \ldots, x_{q(r-d+2)}$. For each i, $1 \leq i \leq r-d+2$, x_{qi} is in the minimal vertex cover C. Then there is a hyperedge e_i covered only by x_{qi}. That is, $e_i \cap C = \{x_{qi}\}$.

Suppose that $\mathfrak{e}_i = e_i \backslash \{x_{qi}\}$. Then the sets $\mathfrak{e}_i$ are submaximal edges such that

$$\mathfrak{e}_1 \sim x_{qi_1}, \mathfrak{e}_2 \sim x_{qi_2}, \ldots, \mathfrak{e}_{r-d+2} \sim x_{qi_{r-d+2}}$$

and $\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \ldots \cup \mathfrak{e}_{r-d+2}$ dose not intersects C. But by hypothesis

$$\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \ldots \cup \mathfrak{e}_{r-d+2}$$

is not independent. That is, it contains a hypergraph e which is not covered by C, a contradiction. □

The following theorem of Villarreal on unmixedness of bipartite graphs can be concluded from the Theorem 3.3, where $r = 2, d = 2$.

Corollary 3.4. [11, Theorem 1.1] *Let G be a bipartite graph without isolated vertices. Then G is unmixed if and only if there is a bipartition $V_1 = \{x_1, \ldots, x_g\}, V_2 = \{y_1, \ldots, y_g\}$ of G such that: (a) $\{x_i, y_i\} \in E(G)$, for all i, and (b) if $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are in $E(G)$, and i, j, k are distinct, then $\{x_i, y_k\} \in E(G)$.*

The following theorem can be concluded from theorem 3.3, where d=2.

Corollary 3.5. [5, Theorem 2.3] *Let G be an r-partite graph which satisfies the condition $(*)$ for r. Then G is unmixed if and only if the following condition hold:*
For every $1 \leq q \leq n$, if there is a set $\{x_{k_1s_1}, \ldots, x_{k_rs_r}\}$ such that

$$x_{k_1s_1} \sim x_{q1}, \ldots, x_{k_rs_r} \sim x_{qr},$$

then the set $\{x_{k_1s_1}, \ldots, x_{k_rs_r}\}$ is not independent.

Now we prove two propositions about d-uniform $(d \geq 2)$ hypergraphs by methods used in the proof of Theorem 3.3.

Proposition 3.6. *Let $\mathcal{H}$ be a d-uniform $(d \geq 2)$ hypergraph on vertex set $\{x_{ji} | \ 1 \leq j \leq n, 1 \leq i \leq d\}$ with perfect matching*

$$\{\{x_{j1}, x_{j2}, \ldots, x_{jd}\} | \ 1 \leq j \leq n\}.$$

If $\mathcal{H}$ is unmixed and has a minimal vertex cover of size n, then for every $1 \leq q \leq n$, if $\mathfrak{e}_1, \mathfrak{e}_2$ be two submaximal edges such that

$$\mathfrak{e}_1 \sim x_{qi_1}, \mathfrak{e}_2 \sim x_{qi_2}$$

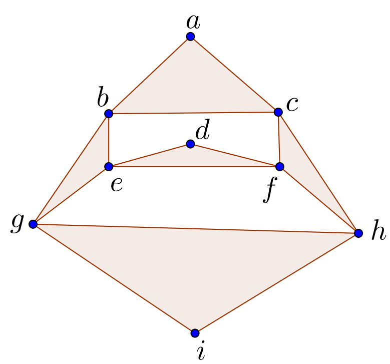
where i_1, i_2 are distinct, then the set $\mathfrak{e}_1 \cup \mathfrak{e}_2$ is not independent.

Proof. Let $1 \leq q \leq n$ be arbitrary. Let in contrary the set $\mathfrak{e}_1 \cup \mathfrak{e}_2$ is independent . Therefore $\mathfrak{e}_1 \cup \mathfrak{e}_2$ is contained in a maximal independent set M. Set $T = V(G) \setminus M$. T is a minimal vertex cover and since it dose not contain any element of $\mathfrak{e}_1 \cup \mathfrak{e}_2$, then T contains x_{qi_1}, x_{qi_2} and then T is at least of size $n+1$, a contradiction. $\square$

Example 3.7. In 3-uniform hypergraph

$$\mathcal{H} = \{\{a,b,c\},\{d,e,f\},\{g,h,i\},\{b,g,e\},\{c,f,h\}\},$$

we have the perfect matching $\{\{a,b,c\},\{d,e,f\},\{g,h,i\}\}$ of size 3.



We show by proposition 3.7 that $\mathcal{H}$ is not unmixed.

Let $\mathcal{H}$ be unmixed (by contrary). $\mathcal{H}$ has the minimal vertex cover $\{b,e,h\}$ of size 3. Now we have 2 hyperedges $\{a,b,c\}$ and $\{b,g,e\}$ in relevence with the hyperedge $\{a,b,c\}$ of perfect matching, but $\{g,e,h,f\}$ is independent, a contradiction.

Proposition 3.8. *Let $\mathcal{H}$ is a d-uniform $(d \geq 2)$ hypergraph on the vertex set $\{x_{ji} | \ 1 \leq j \leq n, 1 \leq i \leq d\}$with perfect matching*

$$\{\{x_{j1}, x_{j2}, \ldots, x_{jd}\} | \ 1 \leq j \leq n\}.$$

Then a sufficient condition for unmixedness of $\mathcal{H}$ is that for every $1 \leq q \leq n$, if $\mathfrak{e}_1, \mathfrak{e}_2$ be two submaximal edges such that

$$\mathfrak{e}_1 \sim x_{qi_1}, \mathfrak{e}_2 \sim x_{qi_2}$$

where i_1, i_2 are distinct, then $\mathfrak{e}_1 \cup \mathfrak{e}_2$ is not independent.

Proof. Let $\mathcal{H}$ satisfies the above condition. We show that $\mathcal{H}$ is unmixed. It is enough to show that every minimal vertex cover of $\mathcal{H}$ contains exactly one element of each hyperedge of the perfect matching. Let

T be a minimal vertex cover. T contains at least one element of each hyperedge of perfect matching. Suppose in contrary that T chooses at least two elements from hyperedge $\{x_{q1}, x_{q2} \ldots, x_{qd}\}$. Without loss of generality, let the elements x_{q1} and x_{q2} are chosen. Since x_{q1} is in minimal vertex cover, then there exist at least $d-1$ distinct vertices in $N(x_{q1})$, such that they dose not belong to T and form a hyperedge together with x_{q1}. Name the set of these vertices $\mathfrak{e}_1$. We have

$$\mathfrak{e}_1 \sim x_{q1}.$$

Similarly, Since x_{q2} is in minimal vertex cover, with a similar argument, there is a submaximal edge $\mathfrak{e}_2$ consisting of $d-1$ distinct vertices, no one belonging to T, such that

$$\mathfrak{e}_2 \sim x_{q2}.$$

Now $\mathfrak{e}_1 \cup \mathfrak{e}_2$ dose not intersect T. But according to hypothesis $\mathfrak{e}_1 \cup \mathfrak{e}_2$ is not independent. That is, it contains a hyperedge e which is not covered by T, a contradiction. □

4. **Edge ideal of unmixed hypergraphs**

In this section, we provide an algebraic interpretation for Theorem 3.3.

Definition 4.1. *Let $\mathcal{H}$ be a hypergraph with $V(\mathcal{H}) = \{x_1, \ldots, x_m\}$. Let $K[x_1, \ldots, x_m]$ be the polynomial ring with indeterminates $x_1, \ldots, x_m$ and coefficients in a field K. For a subset $D = \{x_{i_1}, \ldots, x_{i_r}\} \subseteq V(\mathcal{H})$, let $X_D = x_{i_1} \ldots x_{i_r}$. We define the edge ideal of $\mathcal{H}$ to be*

$$I(\mathcal{H}) := (X_e |\ e \in E(\mathcal{H})).$$

The quotient ring $K[\mathcal{H}] := \dfrac{K[x_1, \ldots, x_m]}{I(\mathcal{H})}$ *is called the edge ring of $\mathcal{H}$.*

Let R be a commutative ring. An element $a \in R$ is called zero divisor if there is $b \neq 0$ in R such that $ab = 0$.

Theorem 4.2. *Let $\mathcal{H}$ be a d-uniform r-partite ($2 \leq d \leq r$) hypergraph which satisfies the condition $(**)$ for $r \geq 2$. Then $\mathcal{H}$ is unmixed if and only if for every $1 \leq q \leq n$, and every $1 \leq i_1 < i_2 < \ldots < i_{r-d+2} \leq r$, $\overline{x}_{qi_1} + \overline{x}_{qi_2} + \ldots + \overline{x}_{qi_{r-d+2}}$ is not a zero divisor in $K[\mathcal{H}]$. Here $\overline{x}_{qi_t}$ denotes the image of x_{qi_t} in $K[\mathcal{H}]$.*

Proof. Let $\mathcal{H}$ be unmixed. If for some $1 \leq q \leq n$ and some $1 \leq i_1 < i_2 < \ldots < i_{r-d+2} \leq r$, $\overline{x}_{qi_1} + \overline{x}_{qi_2} + \ldots + \overline{x}_{qi_{r-d+2}}$ is zero divisor in $K[\mathcal{H}]$, then there is a polynomial $f \notin I(\mathcal{H})$ in

$$S = K[x_{11}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2}, \ldots, x_{1r}, \ldots, x_{nr}]$$

such that $f.(x_{qi_1} + x_{qi_2} + \ldots + x_{qi_{r-d+2}}) \in I(\mathcal{H})$. The ideal $I(\mathcal{H})$ is a monomial ideal and therefore we may assume that f is a monomial and then each monomial of must belong to $I(\mathcal{H})$ (see [3]). That is, each monomial of the above polynomial must be divided by some generator of $I(\mathcal{H})$ which comes from a hyperedge. Let fx_{qi_t} be such a monomial. Then there is a hyperedge e_t in $\mathcal{H}$ such that $X_{e_t}|fx_{qi_t}$. But $X_{e_t} \nmid f$. Then $x_{qi_t}|X_{e_t}$ and $e_t \setminus \{x_{qi_t}\}$ is a subminimal edge $\mathfrak{e}_t$ and $X_{\mathfrak{e}_t}|f$. Therefore, for $x_{qi_1}, x_{qi_2}, \ldots, x_{qi_{r-d+2}}$, There are $r-d+2$ submaximal edges $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_{r-d+2}$ such that $X_{\mathfrak{e}_t}|f$, for $1 \leq t \leq r-d+2$, and

$$\mathfrak{e}_1 \sim x_{qi_1}, \mathfrak{e}_2 \sim x_{qi_2}, \ldots, \mathfrak{e}_{r-d+2} \sim x_{qi_{r-d+2}}.$$

Now by theorem 3.3, the set

$$\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \ldots \cup \mathfrak{e}_{r-d+2}$$

contains a hyperedge e. Now $X_{\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \ldots \cup \mathfrak{e}_{r-d+2}}|f$. Then $X_e|f$. Then $f \in I(\mathcal{H})$, a contradiction.

Conversely, let for for every $1 \leq q \leq n$, and every $1 \leq i_1 < i_2 < \ldots < i_{r-d+2} \leq r$, $\overline{x}_{qi_1} + \overline{x}_{qi_2} + \ldots + \overline{x}_{qi_{r-d+2}}$ is not zero divisor in $K[\mathcal{H}]$. If $\mathcal{H}$ is not unmixed, by theorem 3.3, there is an integer $1 \leq q \leq n$, and submaximal edges $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_{r-d+2}$, such that

$$\mathfrak{e}_1 \sim x_{qi_1}, \mathfrak{e}_2 \sim x_{qi_2}, \ldots, \mathfrak{e}_{r-d+2} \sim x_{qi_{r-d+2}}.$$

where where $i_1, i_2 \ldots, i_{r-d+2}$ are distinct and

$$\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \ldots \cup \mathfrak{e}_{r-d+2}$$

is an independent set. Set $e_t = \mathfrak{e}_t \cup x_{qi_t}$, for $1 \leq t \leq r-d+2$. e_t's are hyperedge. Let $X = X_{\mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \ldots \cup \mathfrak{e}_{r-d+2}}$. X is not in $I(\mathcal{H})$ but $X.(x_{qi_1} + x_{qi_2} + \ldots + x_{qi_{r-d+2}}) \in I(\mathcal{H})$, a contradiction. □

References

[1] C. Berge, *Hypergraphs*, Elsevier Science Publishers B. V., 1989.

[2] H. Haghighi, A generalization of Villarreal's result for unmixed tripartite graphs, *Bull. Iranian Math. Soc.*, **40** (2014), no. 6, 1505-1514.

[3] J. Herzog and T. Hibi, *Monomial Ideals.* Springer, 2010.

[4] J. Herzog, T. Hibi, H. Ohsugi, Unmixed bipartite graphs and sublattices of Boolian lattices, *J. Algebraic Combin.*, **30** (2009), no. 4, 415-420.

[5] R. Jafarpour-Golzari and R. Zaare-Nahandi, Unmixed r-partire graphs, to appear in Bull. Iranian Math. Soc., arXiv: 1511.00228v1 [math. CO].

[6] D. Kiani and S. Saeedi Madani, The edge ideal of complete multipartite hypergraphs, arXiv: 1402.2469v1 [math. CO].

[7] S. Morey, E. Reyes and R. H. Villarreal, Cohen-Macaulay, shellable and unmixed clutters with a perfect matching of König type, *J. Pure Appl. Algebra*, **212** (2008), no. 7, 1770-1786.

[8] G. Ravindra, Well-covered graphs, *J. Combin. Inform.*, System Sci. **2** (1977), no. 1, 20-21.

[9] R. Stanly, *Combinatorics and Commutative Algebra*, 2nd Ed., Progress in Math., Birkhauser, 1996.

[10] R. H. Villarreal, *Monomial Algebras*, Marcel Dekker, Inc., New York, 2001.

[11] R. H. Villarreal, Unmixed bipartite graphs, *Rev. Colombiana Mat.*, **41** (2007), no. 2, 393-395.

[12] D. West, *Introduction to graph theory*, 2nd Ed., Prentic Hall, Upper Saddle River, NJ, 2001.

[13] R. Zaree-Nahandi, Pure simplicial complexes and well-covered graphs, *Rocky Mountain Journal of Mathematics*, **45** (2015), no. 2, 695-702.

Department of Mathematics, Institute for Advanced Studies in Basic Science (IASBS), P.O.Box 45195-1159, Zanjan, Iran

E-mail address: r.golzary@iasbs.ac.ir

Department of Mathematics, Institute for Advanced Studies in Basic Science (IASBS), P.O.Box 45195-1159, Zanjan, Iran

E-mail address: rashidzn@iasbs.ac.ir