THE K&E LOG LOG DUPLEX VECTOR SLIDE RULE

Prelude

This is a TEXed version of *A Supplementary Manual* written by M. P. Weinbach for Keuffel & Esser Log Log Duplex Vector Slide Rule No. 4083, sans the illustrative applications and only include selected examples for brevity, with minor updates to be consistent with 1955 edition of Log Log Duplex Decitrig Slide Rule manual for the content of the supplementary never updated, and additional materials from other slide rule resources to cover aspects of complex valued functions that neither the Decitrig instruction manual nor the original supplementary manual covered.

The K&E Log Log Duplex Vector Slide Rule has all of the scales of the No. 4081 Log Log Duplex Decitrig Slide Rule with scales L and K replaced by scales for hyperbolic functions. The study of the No. 4081 manual is required for basic operations and understanding of this study material.

The original notice about the name change of ST to SRT scale from preface of the 1956 version manual is reproduced below.

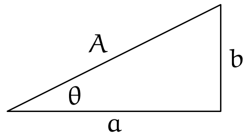
SRT Replaces ST Scale

Note that all references in the following pages to the ST scale should now read SRT, since the SRT scale has replaced the ST on the Log Log Duplex Vector and Log Log Duplex Decitrig Slide Rules.

The SRT scale is of great value, because radians are being used increasingly in modern mathematics and especially in electrical and electronic work. Radians for angles of any magnitude can be read *directly* with the SRT scale. For since and tangent functions of small angles, the SRT scale operates in the exactly the same way as the former ST scale.

I. Plane Vector Calculations

Plane vectors of the form $Ae^{j\theta} = A\angle\theta$, when subject to processes of addition or subtraction, must be converted into their horizontal and vertical components, jointly expressed in complex notation of the form $a + jb$. The quantity a is the horizontal component, b is the vertical component and the symbol $j = \sqrt{-1}$, called "operator", indicates that the component b is 90° in counter clockwise direction form the horizontal component a as shown in the figure.



The exponential form of vectors lands itself more conveniently to the processes of multiplication and division, involution and evolution. Hence when such calculations are be performed on complex numbers of the form $a + jb$, it is necessary to convert them first into the exponential form.

The following rules of conversion are based upon the trigonometric relationship which exists between the components of the vector forming the sides of a right triangle, and the hypotenuse or absolute value of the vector.

A. Conversion of $a + jb$ into $A\angle\theta$

Rule: Set an index of the slide to the larger side a (or b) on scale D. Opposite the shorter slide b (or a) on scale D read angle θ on scale T. Move slide so as to bring angle θ on scale S opposite the shorter side, and at the slide index read A on scale D.

If side a is the larger side, angle θ is smaller than 45°; if side a is the smaller side, angle θ is greater than 45°. This should be taken into consideration when reading angle θ on scale T. In the first case the black numbering of scale T is used, whereas in the second case the red numbering of scale T is used.

B. Conversion of $A\angle\theta$ into $a + jb$

Given $Ae^{j\theta} = A\angle\theta$, find $a + jb$. The complex expression for $A\angle\theta$ is $(A\cos\theta + jA\sin\theta)$. Since scale D gives $\sin\theta$ for values of θ on scale S (black), and $\cos\theta$ for values of θ on scale S (red), the above notation can be evaluated by one setting of the scales S and D.

Rule: To A on scale D set an index of the slide. At e on scale S (black) read the vertical, or imaginary component, and at θ on scale S (red) read the horizontal or real component.

II. Hyperbolic Functions

A. The hyperbolic sine and tangent scales

1. Since for all piratical purposes $\sinh x = x$, and $\tanh x = x$, when $x < 0.1$, scale D gives the $\sinh x$ and $\tanh x$ directly for any value of $x < 0.1$.

2. Scale D in conjunction with scale Sh1 gives the hyperbolic sines of hyperbolic angles x for any value between $0.1 < x < 0.882$. Thus for $x = 0.515$ set indicator on 0.515 scale Sh1 and read $0.538 = \sinh 0.515$ on scale D.

Similarly:

$$\begin{aligned} \sinh 0.645 &= 0.69 \\ \sinh 0.1946 &= 0.1956 \\ \sinh 0.273 &= 0.276 \end{aligned}$$

Conversely, scale Sh1 gives the hyperbolic angle x for $\sinh x = 0.758$ on D, read $x = 0.7$ on Sh1 scale. For $\sinh x = 0.249$ on D, read $x = 0.2466$ on Sh1.

3. Scale Sh2 in conjunction with scale D gives the hyperbolic angle x for values of $\sinh x$ between values of 1 and 10 as read on scale D. Thus for $1.592 = \sinh x$ as read on scale D, we get $x = 1.245$ on scale Sh2. Conversely, scale D gives the corresponding values of $\sinh x$ for values of x between 0.882 and 3.0 on scale Sh2. Thus for $x = 1.245$ on scale Sh2, read $\sinh x = 1.592$ on scale D.

Similarly:

$$\begin{aligned} \sinh 1.465 &= 2.05 \\ \sinh 1.875 &= 3.18 \\ \sinh 2.24 &= 4.64 \\ \sinh 2.95 &= 9.53 \end{aligned}$$

For:

$\sinh x = 5.1$ on D read $x = 2.332$ on Sh2.
$\sinh x = 4.35$ on D read $x = 2.176$ on Sh2.
$\sinh x = 1.84$ on D read $x = 2.37$ on Sh2.

4. Scale Th gives the hyperbolic angle x for values of $\tanh x$ between 0.1 and 1.0, and conversely scale D gives the values of $\tanh x$ for values of x between 0.1 and 3.0. Thus for $x = 0.175$ on scale Th, we read 0.1732 on scale D. For values of x larger than 3.0, the hyperbolic tangent is substantially equal to 1.0.

Similarly:

$$\begin{aligned}\tanh 0.224 &= 0.2202\\ \tanh 0.435 &= 0.409\\ \tanh 0.94 &= 0.735\\ \tanh 1.45 &= 0.895\\ \tanh 2. &= 0.964\\ \tanh 3. &= 0.995\end{aligned}$$

For: $\tanh x = 0.795$ on D read $x = 1.085$ on Th.
$\tanh x = 0.52$ on D read $x = 0.576$ on Th.
$\tanh x = 0.137$ on D read $x = 0.138$ on Th.

B. Evaluation of the hyperbolic cosine

The hyperbolic cosine of any real number x, may be obtained by any of the following formulas:

$$\cosh x = \frac{\sinh x}{\tanh x} \tag{1}$$

$$= \sqrt{1+\sinh^2 x} \tag{2}$$

$$\text{or} \quad = \frac{\sinh x}{\sin\left[\tan^{-1}(\sinh x)\right]} \tag{3}$$

Formula (1) is best suited for slide rule calculation, and the value of $\cosh x$, can be obtained at a single setting of the slide.

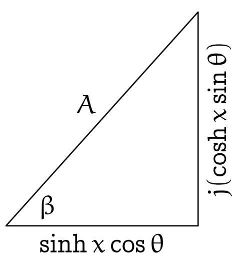
Rule: Set an index of the slide to x on scale Th. Move indicator to x on scale Sh1 or Sh2 and read $\cosh x$ on scale C.

Refer Appendix F for an alternative method computes hyperbolic functions using Log Log Scales.

III. Hyperbolic Functions of Complex Numbers

A. Vector equivalents of $\sinh(x + j\theta)$

The hyperbolic sine of a complex number is a complex number.



Thus

$$\sinh(x + j\theta) = \sinh x \cos\theta + j \cosh x \sin\theta. \tag{1}$$

The vector value of this expression is, therefore

$$\sinh(x + j\theta) = A\angle\beta. \tag{2}$$

where

$$A = \frac{\sinh x \cos\theta}{\cos\beta} \tag{3}$$

or

$$A = \sqrt{\sinh^2 x + \sin^2\theta} \tag{4}$$

and

$$\beta = \tan^{-1}\left(\frac{\cosh x \sin\theta}{\sinh x \cos\theta}\right) = \tan^{-1}\left(\frac{\tan\theta}{\tanh\theta}\right) \tag{5}$$

$$= \tan^{-1}\left(\frac{1}{\tanh x/\tan\theta}\right) \tag{6}$$

$$= \cot^{-1}\left(\frac{\tanh x}{\tan\theta}\right) \tag{7}$$

$$= \cot^{-1}(\tanh x \cos\theta) \tag{8}$$

The numerical value of A may be determined through the relations given either by equation (3) or (4). Any of the last four equations may be used to determine the angle β of the function either directly, or through slide rule operation.

Since the value of the angle θ which enters into all the above expressions, is frequently stated in terms of radians when used in conjunction with hyperbolic function of complex numbers, it is essential that this angle be first converted into degrees. The conversions may be performed easily and quickly by setting 180 on CF to pi on DF. (See Appendix C for using CF and DF for the conversion. Pages 65 to 67 in the 1955 edition Manual of the Log Log Duplex Decitrig Slide Rule explains using SRT scale for conversion).

1. *Slide Rule Method for Calculating the Numerical Value of* $\sinh(x + j\theta)$

Equation (3) indicates that the numerical value of $\sinh(x + j\theta)$ can be obtained easily, provided that the angle β of the function is determined first. The operation consists of a simple multiplication and division with the use of the Sh, S (red), and D scales as follows:

> To x on scale Sh1 or Sh2 as required, set β on scale S (red). Opposite θ on scale S (red), read A on scale D.

If in doing this, the angle θ is beyond the index of the rule body, exchange slide indexes and read A on D, opposite θ on S (red).

Equation (4) indicates that the numerical value of A can also be thought of as the hypotenuse of a right triangle whose sides are $\sinh x$ and $\sin\theta$.

For slide rule operation it may be written

$$A = \sinh x + j\sin\theta \tag{4a}$$

The numerical value of the function may thus be calculated in a manner similar to that of the plane vector as explained in section I. This method does not demand that the angle β of $\sinh(x + j\theta)$ be calculated first. It has the further advantage that it can be remembered easily. We must bear in mind, however, that the auxiliary angle that enters into the calculation of A by this method, is *not* the angle of the vector A.

Whichever method is used to determine the value of A, it is evident from equation (4) that *its value is always larger than the largest of the two terms under the radical.* Thus for $\sinh(0.256 + j10.5^\circ)$, for instance, the value of A is larger than $\sinh 0.256 = 0.259$, while for $\sinh(0.256 + j30^\circ)$, the value of A is larger than $\sin 30^\circ$ or 0.5.

It appears from that has just been said that a casual comparison of the magnitudes of the two terms under the radical of equation (4) is useful in locating the decimal in the reading of the value of A on scale D.

2. Slide Rule Method for Calculating the Angle β *of* $\sinh(x+j\theta)$

Equation (5) indicates that we may think of $\tan\theta$ and $\tanh x$, in their relationship to the angle β, as the sides of a right triangle with β as the angle between one of the sides and the hypotenuse of the triangle:

$$\angle\beta = \tanh x + j\tan\theta \tag{5a}$$

It should be fully realized that in equation (5a) we are concerned only with the angle β, and *not* in the numerical value of the apparent vector relation between the $\tanh x$ and $\tan\theta$.

From what has just been said, it follows that the slide rule method for obtaining the value of the angle β of $\sinh(x+j\theta)$ is fundamentally identical to that of obtaining the angle of a plane vector as explained in section I provided that, instead of x and θ, $\tanh x$ and $\tan\theta$ were known. Thus we may obtain the angle β for all values of x and θ by first obtaining the values of a and b corresponding to $\tanh x$ and $\tan\theta$ respectively, on scale D. Set slide index on the larger component on scale D, and read the angle β on scale T over the smaller component. This is the most direct method for obtaining the angle β.

Example 1. Evaluate β of $\sinh(0.256 + j10.5^\circ)$.

Solution:
$$\begin{aligned} \tanh 0.256 &= 0.25 \\ \tan 10.5^\circ &= 0.185 \end{aligned}$$

To 0.25 on scale D set right slide index. Move indicator to 0.185 on scale D and read $\beta = 36.5^\circ$ on scale T (black).

Very definite and simple rules which eliminate the necessity of ascertaining first the values of $\tanh x$ and $\tan\theta$, and based upon the above method are formulated below.

It should be observed from equation (3) in its relation to the tangent scales ST, T (black) and T (red), that the angle β of the sinh function may be smaller than 5.75°, in which case it must be read on scale ST; it may also be between 5.75° and 45° on scale T (black); or it may be larger then 45° on scale T (red). Furthermore, the value of angle θ may also be on any of these three scales. It is therefore important to determine a method for ascertaining the scale on which the angle β should be read.

The following cases may occur:

Case 1. $\theta < 5.75^\circ$; $\beta \lesseqgtr 5.75^\circ$.

Since $\tanh x < 1$ and $0.01 < \tan\theta < 0.1$, it follows that $0.01 < \tan\beta < 1$. The angle β may be, therefore either smaller or greater than 5.75°, and thus lie either on scale SRT or on scale T (black). It is just equal to 5.75° when $\tan\theta = 0.1\tanh x$. Hence *with slide and body matched,* $\tan\theta < 0.1\tanh x$ *when* θ *on scale SRT,* is to the left of x on scale Th, the angle $\beta < 5.75^\circ$ should be read on the scale SRT. On the other hand, *when* θ *is to the right of* x *on scale Th,* $\tan\theta > 0.1\tanh x$, and *the angle* $\beta > 5.75^\circ$ *should be read on scale T (black).*

Another way of stating this is as follows:

Set indicator over x on scale Th, and move θ on scale SRT under indicator. If slide protrudes to the right, $\beta < 5.75^\circ$, and should be read on scale SRT. If the slide protrudes to the left, $\beta > 5.75°$, and should be read on scale T (black).

To determine the value of the angle β.

To x on scale Th, set θ on scale SRT. At body index read β on SRT, if slide protrudes to the right; or on scale T (black), if slide protrudes to the left.

Example 2. Evaluate $\sinh(0.216 + j3.5°)$.

To find the angle β: Note that $\theta = 3.5$ is to the right of 0.216 on scale Th. To $x = 0.216$ on scale Th set $\theta = 3.5°$ on scale SRT. At body index read $\beta = 16°$ on T (black).

To find the numerical value of the function: To $x = 0.216$ on scale Sh1 set $\beta = 16°$ on scale S (red). Opposite 3.5° on scale S (red), read $A = 0.226$ on scale D. Hence $\sinh(0.216 + j3.5°) = 0.226\angle 16°$.

Example 3. Evaluate $\sinh(0.6 + j2.5°)$.

To find the angle β: Note that 2.5 on scale ST is to the left of 0.6 on scale Th. To $x = 0.6$ on scale Th set $\theta = 2.5$ on scale SRT. At body index read $\beta = 4.66°$ on scale SRT.

To find the numerical value of the function: To $x = 0.6$ on scale Sh1 set $\beta = 4.66°$ on scale B (red). Opposite $\theta = 2.5°$ on scale S (red), read $A = 0.638$ on scale D. Hence $\sinh(0.6 + j2.5°) = 0.638\angle 4.66°$.

The above example shows that for all practical purposes, when both θ and β are less than about 5°, the numerical value of the sinh function of a complex number is substantially equal to $\sinh x$.

Case 2. $5.75° < \theta < 45°$; $\beta \lesseqgtr 45°$.

Since $\tanh x < 1$ and $0.1 < \tan\theta < 1$, it follows that $0.1 < \tan\beta < 10$. The angle β may be, therefore either smaller or greater than 45°, and thus lie either on scale T (black) or on scale T (red). The angle β is just equal to 45° when $\tan\theta = \tanh x$. Hence with slide and body matched, $\tan\theta < \tanh x$ when θ on scale SRT, is on the left of x, on scale Th. Consequently, the angle $\beta < 45°$ should be read on the scale T (black). when angle θ is to the right of x on scale Th, $\tan\theta > \tanh x$, and the angle $\beta > 45°$ should be read on scale T (red).

It follows, therefore, that the rule given under Case 1 holds also for Case 2. If with angle θ on scale T (black) over x on scale Th, the slide protrudes to the right, the angle $\beta < 45°$ should be read on scale T (black); and if the slide protrudes to the left, the angle $\beta > 45°$ should be read on scale T (red).

To determine the value of β, we must first ascertain whether it is smaller or larger than 45°.

(2a) $5.75 < \theta < 45°$ and $\beta < 45°$.

To x on scale Th, set θ on scale T (black). Opposite body index read β on scale T (black).

(2b) $5.75 < \theta < 45°$ and $\beta > 45°$.

To obtain the angle β for this case: with slide and body matched, set indicator over θ on scale T (black), move left slide index under indicator and opposite x on scale Th read β on T (red).

Another method which may be used to advantage is as follows:

> To x on scale Th, set θ o scale T (black). At slide index read $\cot\beta$ on scale D. Opposite this reading on scale C, read angle β on T (red).

Case 3. $\theta > 45°$; $\beta > 45°$.

> Referring to equation (3) it is seen that since $\tanh x < 1$ and $\tan\theta > 1$, the angle of the sinh function will always be larger than 45° and should be read on scale T (red).

The method of obtaining β is:

> To right body index set θ on scale T (red). At x on scale Th read β on scale T (red).

Another method which may be used is as follows:

> Set slide index to x on scale Th. Opposite θ on T (red), read $\cot\beta$ on scale D. Opposite this value of scale C, read β on T (red).

B. Vector equivalents of $\cosh(x + j\theta)$

C. Calculation of the vector value of $\tanh(x + j\theta)$

Since

$$\tanh(x + j\theta) = \frac{\sinh(x + j\theta)}{\cosh(x + j\theta)} \tag{16}$$

It follows from the discussion given in sections (A) and (B) that the hyperbolic tangent of a complex number is a vector quantity of the form

$$\tanh(x + j\theta) = D\angle\delta. \tag{17}$$

where by equation (16)

$$D\angle\delta = \frac{A\angle\beta}{B\angle\alpha} = \frac{A}{B}\angle(\beta - \alpha). \tag{18}$$

From the above it follows that the vector value of the tanh function of a complex number must be obtained by evaluating the sinh and the cosh functions of the same complex numbers as outlined in the preceding two sections, and taking the ratio of the vector value of the sinh to the vector value of the cosh functions.

IV. Inverse Hyperbolic Functions of Complex Numbers

A. Slide rule calculation of $\sinh^{-1}(A\angle\beta)$

To calculate the complex number $x + j\theta$ when the hyperbolic sine of this complex number is given, it is necessary that the vector value of the function be first split up into its components by the method outlined in section I (B). We thus get

$$\sinh(x + j\theta) = A\cos\beta + jA\sin\beta = a + jb \tag{19}$$

where x and θ are the quantities to be calculated.

Appendix A: Addition without Linear Scale

Introductory materials never mention how to handle adding two numbers using slide rule. A method that can use either C & D or CF & DF is recorded here for completeness. This has similarity to the method in Appendix B.

Adding two numbers can be done on slide rule using the principle

$$x + y = x(1 + y/x)$$
$$x - y = y(x/y - 1) \qquad \text{where } y < x$$

Rule: Set the index of C (or CF) on the smaller number a. Moving hairline to the larger number b on D (or DF) scale, read the number b/a on C (or CF) then multiply a by $b/a+1$ or $b/a-1$ and read result on D (or DF), beware of which digit the number 1 is required to add or subtract.

If using C & D, the multiplication can be done without doing another setting.

Appendix B: Impedance and the B Scale

Method used is from Using The Slide Rule In Electronic Technology by E. Charles Alvarez, this is another way to calculate the hypotenuse of the triangle requires only C, D and B scales.

The impedance of a circuit containing a resistance and inductance is given as

$$Z = \sqrt{R^2 + X_L^2}$$

The impedance of a circuit containing a resistance and capacitive reactance is given as

$$Z = \sqrt{R^2 + X_c^2}$$

The C, D, and B scales provide a short-cut answer when it is not necessary to know the angle involved.

Rule: Set the C scale index on the *smaller* of the two numbers, move hairline over the larger number on the D scale, read the number x on B scale and slide hairline over $x+1$ on B, read the answer on D scale.

The above rule computes

$$a\sqrt{1 + (b/a)^2} = a\sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{a^2 + b^2}.$$

If the number under the B scale is in the right half of the rule — for example in finding the impedance of $R = 2$ and $X_L = 7$, — the number under the hairline is read as a two digit number, (i.e. 12.3). Adding 1 and moving the hairline to the 13.3 yields the result on the D scale, 7.28.

Appendix C: Conversions Between Degrees and Radians

Excerpt from section 36 of Dietzgen 1725 Vector Type Log Log Slide Rule Manual written by Ovid W. Eshbach, H. Loren Thompson. The references to "ST" has been replaced by "SRT". The section 39 To change radians to degrees or degrees to radians from 1943 edition of the manual (page 59, 60) has similar content, without giving the error analysis.

If an angle is made a central angle of a circle, the number of radians in the angle equals the ratio of the length of the intercepted arc to the length of the radius of the circle. Hence, since an angle of 180° intercepts an arc equal to a semi-circle,

$$180^\circ = \frac{\pi r}{r} = \pi \text{ radians.}$$

Therefore, the following relation can be set up:

$$\frac{\pi}{180} = \frac{R \quad (\text{number of radians})}{D \quad (\text{number of degrees})}.$$

Based on this proportion we have the following *general rule* for conversions between degrees and radians:

Rule: Set 180 on CF to π right on DF.

> Opposite a given number of degrees on CF (or C) read the equivalent number of radians on DF (or D).
>
> Opposite a given number of radians on DF (or D) read the equivalent number of degrees on CF (or C).

The decimal point ls located from a mental estimate. (1 radian $= \frac{180^\circ}{\pi} = 57.3^\circ$).

Example 1. How many radians are equivalent to 125.5°?

> Set 180 on CF to π right on DF. Opposite 125.5 on CF (or C) read 2.19 radians on DF (or D).

Example 2. How many degrees are equivalent to 5.46 radians?

Set 180 on CF to π right on DF. Opposite 5.46 on D read 313° on C.

If, in conversions between radians and degrees, an accuracy of $\frac{1}{6}$ of 1% is sufficient, then the conversion can be made more simply. For small angles $\sin A = A$ (in radians) to a close approximation. Therefore, opposite an angle marked on the SRT scale, we can read its radian measure $A = \sin A$ on the D scale. For $A = 1°$ the error in the approximation is 1 part in $200,000$. For $A = 5.74°$, the maximum angle on the SRT scale, the error is 1 part in 600, or $\frac{1}{6}$ of 1%. To this accuracy, then angles marked in degrees on the SRT scale have their radian values indicated on the D scale. *Decimal multiples of angles on the SRT scale will have radian values which are decimal multiples of the values read on the D scale.* Hence the following *general rule*:

Rule: To convert between degrees and radians, to an accuracy of $\frac{1}{6}$ of 1% or better, read radians on D opposite degrees on SRT, or vice versa.

Example 3. How many radians are equivalent to 125.5° radians?

Set hairline to 125.5° (or 1.255°) on SRT. Under hairline on D read 2.19 radians.

Example 4. How many degrees are equivalent to 5.46 radians?

Close rule and set hairline of 5.46 on D. Under hairline on SRT read 313°.

Appendix D: Decibel Computations

From 4082-3 The Log Log Duplex Decitrig Slide Rule With F Scale, usable without F scale.

In Radio or Communication engineering, power, current and voltage ratios are often expressed in decibels (db).

For power ratios:

$$\text{db} = 10 \log_{10} \frac{P_1}{P_2}$$

The quickest solution of this formula is obtained with the aid of the log log scales as follows:

To 10 (the base of the common logarithms) on LL3 set 10 (the factor) of the C scale; then opposite any power ratio on the log log scales, the corresponding decibel value can be read on the C scale.

Example 5. How many db correspond to a power ratio $\frac{P_1}{P_2} = 4.36$?

With slide set as explained. Opposite 4.36 on LL3 scale read 6.4 on C scale. (The decimal point is easily placed by remembering that the index, set to 10 on LL3, is 10).

Example 6. $\frac{P_1}{P_2} = 20,000$.

Slide set as explained. Opposite 20,000 on LL3 scale, read 43. on C scale.

Example 7. What is the power ratio corresponding to 20.86 db?

With slide set as explained. Opposite 20.86 on C scale read 122. on LL3 scale.

For voltage or current ratios:

$$\text{db} = 20 \log_{10} \frac{V_1}{V_2} \quad \text{and} \quad \text{db} = 20 \log_{10} \frac{I_1}{I_2}$$

To 10 (te base of the common logarithms) on LL3 set 20 (the factor) of the C scale; then opposite any voltage or current ratio on the log log scales, the corresponding decibel value can be read on the C scale.

Example 8. How many db correspond to a voltage or current ratio of 150?

Slide set as explained. Opposite 150 on LL3 scale read 43.5 on C scale.

Example 9. What is the voltage ratio corresponding to 20.8 db?

With slide set as explained. Opposite 20.8 on C scale read 10.95 on LL3 scale.

Appendix E: Useful Properties for Log Log Scales

To compute x^y when $x > e$ and $y > 10$, rewrite using

$$x^y = (x^a)^b = z^{y/\log_x z} \qquad (y = ab, x^a = z)$$

so that $z = x^a$ can be found on LL1 or LL2 and the subsequent z^b can be computed in one rule set. It is desirable to only regard the result as an approximation, since the exponentiation result can be quite sensitive to subtle change in exponent.

Example 1. Evaluate 3^{84} using $z = 101$. ($3^{84} = 1.197 \times 10^{40}$)

First find $\log_3 101 = 4.20$ using LL3. Then set index to 1.01 on LL0/LL1 and move hairline to 2 on scale C and read 1.22 on scale LL2, for the result $3^{84} = 101^{84/4.2} = 1.01^{20} \times 10^{40} = 1.22 \times 10^{40}$

It is also possible to use fast exponentiation for more accurate result, that for the above example $84 = 2^6 + 2^4 + 2^2$, so $3^{84} = 3^{64} \times 3^{16} \times 3^4$. Use LL3 to find $3^4 = 81$, then use C, D and A scale to find $3^{16} = 81 \times 81^2 = 4.30 \times 10^7$ and $3^{64} = (4.30 \times 10^7)^4 = 3.41 \times 10^{30}$. Use C and D to find the result is $81 \times 4.3 \times 3.41 \times 10^{37} = 1.19 \times 10^{40}$.

To compute $\log_a b$ when $a > 10{,}000$ or $b > 10{,}000$, it is useful to apply the relation

$$\log_a b = \log_{a^x} b^x = x \log_{a^x} b = x^{-1} \log_a b^x \qquad (x \neq 0).$$

A typical value to choose is $x = 0.1$.

Example 2. $\log_{45000} 42$.

Set hairline to 42 on LL3 and read $42^{0.1} = 1.453$ on LL2. Then decompose $45000^{0.1} = 4.5^{0.1} \times 10^{0.4} = 1.162 \times 2.51 = 2.92$. To find $\log_{45000} 42 = \log_{2.92} 1.453$, set left index 2.92 on LL3, opposite 1.45 on LL2 and read result 0.349 on scale C.

An interesting fact related to the usage of LLD scale (see page 96 of Log Log Duplex Decitrig Slide Rule Manual): When $-0.01 < xy < 0.01$, $(1 + x)^y \approx 1 + xy$, also if $1/y = x < 0.01$, $(1 \pm x)^{\pm y} \approx e$, for

$$e = \lim_{x \to \infty} (1 + x^{-1})^x.$$

Appendix F: Hyperbolic Functions by Exponential Formulas

Hyperbolic functions can be evaluated on slide rules that do not have Sh and Th scales by using the following exponential expressions:

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} \approx \frac{e^x}{2} \\ & \qquad\qquad\qquad\qquad \text{when } x > 3 \\ \cosh x &= \frac{e^x + e^{-x}}{2} \approx \frac{e^x}{2} \\ \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \approx 1 \qquad \text{when } x > 3.8 \end{aligned}$$

For $1.0 < x < 3.0$, e^x and e^{-x} can be found at same time on LL3 and LL03, respectively. Other Log Log Scales cover the $0.01 < x < 1.0$

range, and as described in the page 96 of Log Log Duplex Decitrig Slide Rule Manual, D scale can be used as LLD and LL0D for $x < 0.01$.