Formula Sheet Available in Exam

The following formulae will be available in the exam:

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a$, ... ,$x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-Time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
SNR	Signal-to-Noise Ratio

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ or, more generally, $\sum_{n=q}^{r} \alpha^n z^{-n} = \frac{\alpha^q z^{-q}-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$	$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}$
DFT:	$X[k] = \sum_{0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$	$x[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n]\cos\frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N}\sum_{n=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n]*y[n] = \sum_{r=-\infty}^{\infty} x[r]y[n-r] \quad\Leftrightarrow\quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$

$v[n] = x[n]y[n] \quad\Leftrightarrow\quad V\left(e^{j\omega}\right) = \frac{1}{2\pi}X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right)Y\left(e^{j(\omega-\theta)}\right)d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] = \sum_{r=0}^{N-1} x[r]y[(n-r)_{\bmod N}] \quad\Leftrightarrow\quad V[k] = X[k]Y[k]$

$v[n] = x[n]y[n] \quad\Leftrightarrow\quad V[k] = \frac{1}{N}X[k] \circledast_N Y[k] = \frac{1}{N}\sum_{r=0}^{N-1} X[r]Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10}b}{4.6\Delta\omega}$

where a =stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in normalized rad/s.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$