Calculating the Sum of Cubes of Integers from 1 to n

If you are a math student, you probably had at least once to prove the formula for the sum of squares using mathematical induction. Mathematical induction is a good ways to prove such identities provided that you already know them. A more advanced way to get the result is by using generating functions.

The Generating Function

A generating function is a taylor series that uses elements of an infinite sequene as its coefficients. You can calculate those coefficient either by deriving the function, or by applying elementary arithmetical operations.

A very basic generating function is:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$$

Now, it is easy to prove that for a natural number *k*

(1) $$\frac{1}{(1-x)^k}=\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}x^n$$

The Sequence

We are looking for a sequence a_n , such *that* for every non-negative integer *n:*

$$a_n=\sum_{k=0}^{n} k^3$$

Let us define it recursively, so

$$a_0=0$$
$$a_{n+1}=a_n+(n+1)^3$$

Now, let *S* be our generating function $S=\sum_{n=0}^{\infty} a_n$, so:

$$\sum_{n=0}^{\infty} a_{n+1}x^{n+1}=\sum_{n=0}^{\infty} a_n x^{n+1}+\sum_{n=0}^{\infty}(n+1)^3 x^{n+1}$$

which is:

(2) $S-a_0=xS+\sum_{n=0}^{\infty} n^3 x^n$

Now, what is $\sum_{n=0}^{\infty} n^3 x^n$? Fisrt, let's find $\sum_{n=0}^{\infty} nx^n$ fi

$$\frac{1}{(1-x)^2}=\frac{d}{dx}\frac{1}{1-x}=\frac{d}{dx}\sum_{(n=0)}^{\infty} x^n=\sum_{n=0}^{\infty}(n+1)x^n$$

Let us multiply both sides by x, and get:

$$\frac{x}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^{n+1}=\sum_{n=0}^{\infty} nx^n$$

Let us derive once again:

$$\frac{(1-x)^2+2x(1-x)}{(1-x)^4}=\frac{1-2x+x^2+2x-2x^2}{(1-x)^4}=\frac{1-x^2}{(1-x)^4}=\frac{(1-x)(1+x)}{(1-x)^4}=\frac{1+x}{(1-x)^3}=\sum_{n=1}^{\infty}(n^2)x^{n-1}$$

Now, let's multiply both sides by x:

$$\frac{x(x+1)}{(1-x)^3}=\sum_{n=0}^{\infty} n^2 x^n$$

Now, let us derive once again:

$$\frac{(2x+1)(1-x)^3+3(x^2+x)(1-x)^2}{(1-x)^6}=\frac{[(2x+1)(1-x)+3(x^2+x)]}{(1-x)^4}=\frac{x^2+4x+1}{(1-x)^4}=\sum_{n=1}^{\infty} n^3 x^{n-1}$$

Multiply by x:

$$\frac{x^3+4x^2+x}{(1-x)^4}=\sum_{n=0}^{\infty} n^3 x^n$$

Now, let us use this identity in (2):

$$S=xS+\frac{x^3+4x^2+x}{(1-x)^4}$$

$$S(1-x)=\frac{x^3+4x^2+x}{(1-x)^4}$$

$$S=\frac{x^3+4x^2+x}{(1-x)^5}$$

From (1):

$$S=(x^3+4x^2+x)\sum_{n=0}^{\infty}\binom{n+4}{4}x^n=(x^3+4x^2+x)\sum_{(n=0)}^{\infty}\frac{(n+1)(n+2)(n+3)(n+4)x^n}{24}=$$

$$=\sum_{n=0}^{\infty}\frac{(n+1)(n+2)(n+3)(n+4)x^{(n+3)}}{24}+4\sum_{(n=0)}^{\infty}\frac{(n+1)(n+2)(n+3)(n+4)x^{n+2}}{24}+$$

$$+\sum_{n=0}^{\infty}\frac{(n+1)(n+2)(n+3)(n+4)x^{n+1}}{24}=\sum_{n=0}^{\infty}\frac{n(n+1)(n-1)(n-2)x^n}{24}+$$

$$+4\sum_{n=0}^{\infty}\frac{n(n+1)(n-1)(n+2)x^n}{24}+\sum_{n=0}^{\infty}\frac{n(n+1)(n+2)(n+3)x^n}{24}=$$

$$=\sum_{n=0}^{\infty}\frac{n(n+1)[(n-1)(n-2)+4(n-1)(n+2)+(n+2)(n+3)]x^n}{24}=$$

$$=\sum_{n=0}^{\infty}\frac{n(n+1)(n^2-3n+2+4n^2+4n-8+n^2+5n+6)x^n}{24}=\sum_{n=0}^{\infty}\frac{n(n+1)(6n^2+6n)x^n}{24}=$$

$$=\sum_{n=0}^{\infty}\frac{[n(n+1)]^2x^n}{4}$$

Thus,

$$a_n=\frac{[n(n-1)]^2}{4}=\left(\sum_{k=0}^{n}k\right)^2$$